3. Dynamic Games of Complete Information

The extensive form representation of a game

Nodes, information sets

Perfect and imperfect information

Addition of random moves of nature (to model uncertainty not related with decisions of other players).

Mas-Colell, pp. 221-227, full description in page 227.
The Prisoner's dilemma

<table>
<thead>
<tr>
<th>Player 1</th>
<th>confess</th>
<th>don't confess</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 2</td>
<td>confess</td>
<td>2.2</td>
</tr>
<tr>
<td></td>
<td>don't confess</td>
<td>0.10</td>
</tr>
</tbody>
</table>

One extensive form representation of this normal form game is:
Player 1

confess

Player 2

payoffs

confess

Player 2

don't confess

Player 2

don't confess

confess

Player 2

Player 2
Both players decide whether to confess simultaneously (or more exactly, each player decides without any information about the decision of the other player).

This is a game of *imperfect information*: The information set of player 2 contains more than one decision node.
The following extensive form represents the same normal form:
Extensive form game:

Player 1

confess

Player 2

confess

\( \begin{pmatrix} 2 \\ 2 \end{pmatrix} \)

Player 2

don't confess

\( \begin{pmatrix} 0 \\ 10 \end{pmatrix} \)

Player 2

confess

\( \begin{pmatrix} 10 \\ 0 \end{pmatrix} \)

Player 2

don't confess

\( \begin{pmatrix} 6 \\ 6 \end{pmatrix} \)

payoffs
Now player 2 *observes* the behavior of player 1. This is a game of *perfect information*. Strategies of player 2: For instance "If player 1 confesses, I...; if he does not confess, then I..."

Player 2 has 4 strategies: (c, c), (c, d), (d, c), (d, d)

**The normal form of this game is:**

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>c, c</td>
</tr>
<tr>
<td>Player 1</td>
<td>confess</td>
</tr>
<tr>
<td></td>
<td>don't confess</td>
</tr>
</tbody>
</table>

There is only one NE,
- strategy confess for player 1 and
- strategy (c, c) for player 2.
Now player 2 observes the behavior of player 1. Strategies of player 2: For instance "If player 1 confesses, I…; if he does not confess, then I…"

**Backward induction:**

A player’s strategy must specify optimal actions at every point in the game tree. When a player is in a given node, she should play optimally from that point on (given her opponent’s strategy).

We solve a game by backward induction when we first find the optimal behavior at the final nodes of the game and then move up, determining what is the optimal behavior earlier in the game taking into account this later behavior.
Backward induction in the prisoner's dilemma played sequentially:
This procedure makes sense; moreover, it allows us to rule out Nash equilibria that are not reasonable. Consider now the following extensive form game (Mas-Colell, p.269, fig 9.B.1):
Player 2 *observes* the behavior of player 1. This is again a game of *perfect information*.

The normal form of this game is:

<table>
<thead>
<tr>
<th>Player 1</th>
<th>fight</th>
<th>accommodate</th>
</tr>
</thead>
<tbody>
<tr>
<td>do not enter</td>
<td>0.2*</td>
<td>0.2</td>
</tr>
<tr>
<td>enter</td>
<td>-3.-1</td>
<td>2.1*</td>
</tr>
</tbody>
</table>

There are two NE, are both reasonable?

If we look at the extensive form of the game and we solve by backward induction:
There is another nice example in Mas-Colell in page 271 (fig 9.B.3)

*results for finite game of perfect information*

They can be solved by backward induction (Zermelo)

When we solve by backward induction we obtain a Nash-equilibrium

Those games have a pure strategy Nash-equilibrium (at least one), and can be solved by backward induction.

Hence, solving by backward induction we discard, maybe, some NE, but we surely find the reasonable ones. And we always have a “solution” of the interaction modeled by the game
What about finite games with imperfect information?
Example in Mas-Colell, p. 274, fig. 9.B.4
Definition: **subgame of an extensive form**. A subset of a game that:

- Begins in an information set with only one node and
- If a node is in the subgame, every node of its information set is also in the subgame

In the former extensive form, there is a proper subgame
A simultaneous-move subgame within the full game

Player 1

Player 2

\[
\begin{pmatrix}
0 & 1 \\
2 & 3
\end{pmatrix}
\]
Subgame perfect Nash equilibrium

Find a Nash-equilibrium in every subgame backwards.

The normal form of the subgame is:

<table>
<thead>
<tr>
<th></th>
<th>fight</th>
<th>accommodate</th>
</tr>
</thead>
<tbody>
<tr>
<td>fight</td>
<td>-3,-1</td>
<td>1,-2</td>
</tr>
<tr>
<td>acco</td>
<td>-2,-1</td>
<td>3,1*</td>
</tr>
</tbody>
</table>

The subgame has only one Nash-equilibrium. Then player 1 compares do not enter (he obtain a payoff of zero) and the payoff in the expected Nash-equilibrium if he enters (he expects a payoff of 3). Hence he enters.

An example with continuous payoffs. A Stackelberg game.
Sequential game: Firm 1 chooses $q_1$, firm 2 observes this choice and chooses $q_2$:

$$R_2(q_1) = \begin{cases} 
\frac{1}{2}(A - q_1 - c) & \text{whenever } q_1 < A - c \\
0 & \text{otherwise}
\end{cases}$$

$$\max_{q_1} \pi_1(q_1, R_2(q_1)) = (A - q_1 - R_2(q_1)) q_1 - cq_1, \text{ s.t.: } q_1 \geq 0$$

$$q_1 = \frac{1}{2}(A - c)$$

$$q_2 = \frac{1}{4}(A - c)$$

Payoffs in the different situations?
Is it always compelling the backward induction argument? The centipede game.
**Forward induction**
Example in Brandenburger’s Notes (see Yildiz also)
The normal form of the subgame is:

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>fight</td>
<td>fight</td>
</tr>
<tr>
<td>0,0*</td>
<td>2,-1</td>
</tr>
<tr>
<td>accommodate</td>
<td>accommodate</td>
</tr>
<tr>
<td>-1,2</td>
<td>4,4*</td>
</tr>
</tbody>
</table>