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PART II LABOUR SUPPLY

9 On the specification of labour supply functions

NICHOLAS STERN*

9.1 Introduction

There are many criteria which might be relevant in choosing a functional form for a labour supply function: consistency with utility theory; convenience in estimation; facility for incorporation in theoretical studies; ease of use in applied problems (e.g. the analysis of tax reform); flexibility in the type of response it permits, and so on. The relative importance of these considerations will depend on the application one has in mind. In this paper we set out and discuss various criteria and examine a number of functional forms which are commonly used to see how far they meet the yardsticks described. We shall be concerned mostly with the neo-classical theory of labour supply in its simplest form: one type of labour, utility maximisation, and the absence of rationing. We shall conclude that a single form should be used only with great caution, and recommend diversity. Presumably this also applies to more complicated models.

Many of the criteria we shall use have been discussed intensively in the literature. However, some of them have received insufficient attention and there does not appear to have been any attempt to assemble them so that one may judge the pros and cons of various forms. Certain of the literature surveys (see, e.g., Killingsworth, 1983, for a comprehensive account) indicate, with some important exceptions, an absence of concern with the properties of the function selected. We shall lay special emphasis on two features: the flexibility of response permitted by a function and its use in the analysis of tax reform. It is well understood in standard consumer demand theory that certain functional forms for utility and demand functions force special features in responses that are likely to be inconsistent with the data (see, e.g., Deaton, 1974, on additive separability). Further, in this context the choice of functional form may have profound consequences for policy analysis–for example, in some models of optimum taxation certain policies (such as uniform proportional taxes) may be entirely a consequence of the form chosen and independent of whichever parameters happen to be estimated (see e.g. Atkinson 1977, Deaton 1981, and Atkinson, Stern and Gomulka 1980). Since the standard theory and estimation of labour supply is to a large extent an application of demand theory, it is natural to pose the same type of questions for labour supply functions as one would for demand functions. In part these questions can be investigated using pure theory and in part by examining the sensitivity of policy judgements to functional form in applied work. This paper is intended as a prelude to both types of enquiry.

In the next section we shall set out the criteria and discuss how they can be used to appraise features of labour supply functions. One can in general start from the labour supply function and derive the utility function or start from the utility function and calculate the labour supply function. In Section 9.3 we consider forms which arise from either route but in each case a natural requirement is that both the utility function (direct or indirect) and the labour supply function should be reasonably tractable. We shall indicate the availability of a class of labour supply functions analogous to those which are very often used but which appear to have received insufficient attention. However, we shall not attempt to be exhaustive in our choice of functional forms in Section 9.3—our purpose is to show how particular forms can be checked against the criteria and to carry out the checks for some common examples. In Section 9.4 we assess how the functions have fared against our criteria and Section 9.5 contains concluding remarks. Tables 9.2–9.14 are at the end of the paper.

9.2 The criteria

Our criteria for labour supply functions are set out in summary form in Table 9.1. It should be reasonably clear that each of them will be relevant in certain circumstances but also that there are few contexts in which we would want to, or be able to, insist that all of them should apply. And all of them have been discussed at various points in the literature (some references will be given below), although some more intensively than others.

The first two, and particularly the second, have been the most prominent. The recent voluminous literature on labour supply (see Killingsworth, 1983, for a useful survey) has been particularly concerned with the problems of estimation and thus we shall not be emphasising this aspect here. More recently (see e.g. Burtless and Hausman, 1978, Hausman, 1980 and 1981*a* and *b*; Deaton and Muellbauer, 1981) the third criterion, the relation between the labour supply function and the utility function, has been emphasized. This has close links with the problem of estimation with a non-linear budget constraint (and thus with criterion (2)) and with applied policy problems where explicit judgments on changes in household welfare are required.

Table 9.1. *Criteria*

(1)	consistency with utility maximisation—in particular the Slutsky condition;
(2)	convenience in estimation: (i) linearity in coefficients, (ii) incorporation of household characteristics, (iii) stochastic variation;
(3)	ease of calculation of direct and indirect utility functions, the expenditure function and the inverse supply function;
(4)	ease of use in applied policy problems: largely criteria 2 and 3 above, together with transparency of the important parameters;
(5)	facility of computation in optimum income tax models (and, in particular, additive separability);
(6)	behaviour of labour supply at low levels of work: (i) the possibility of negative marginal disutility of labour, (ii) the possibility that leisure might be inferior;
(7)	aggregation;
(8)	flexibility in possible response of labour to changes in the wage.

Tractability of forms is important not only in applied problems but also in theory and, particularly in the case of labour supply, in problems of income taxation, criterion (5). The forms should not only be tractable but we should be aware of the consequences of the choice of form for the result. In both optimum income taxation and in estimation, the behaviour of the labour supply function and the utility function at low or zero levels of labour supply are important. In the former case, because this will determine the extent of unemployment associated with particular tax functions and, in the latter, because the decision whether or not to work plays a key role in estimation. Thus criterion (6) will be of substance. Given that labour supply data are often in aggregated form we include the possibility of aggregation as the seventh of the criteria.

The final criterion is the one which originally motivated the paper. Many applied studies use forms such as the linear which allow very little flexibility. It is possible that the slope and curvature of labour supply functions vary considerably over relevant ranges of wages and incomes, and it would be unfortunate if the forms that we use forced the estimated responses to be tightly restricted. The result of so doing may have important consequences for both estimation and policy.

In the remainder of this section we set out and discuss the criteria in turn and indicate how a particular labour supply function may be checked against them.

1. *Consistency with utility maximisation*

In the literature on demand theory, the general 'integrability' problem with n goods is as follows. Given a set of demand functions $\mathbf{x}(\mathbf{p}, m)$, where $\mathbf{x}$ is the vector of quantities, $\mathbf{p}$ is the price vector, and m is lump-sum income, find a utility index $u(\mathbf{x})$ so that the maximisation of $u(\mathbf{x})$ subject to the constraint $\mathbf{px} \leqslant m$ yields as a solution the demand functions $\mathbf{x}(\mathbf{p}, m)$. Sufficient conditions for a solution were indicated by Samuelson (1947, p. 116) and a proof provided in Samuelson (1950). A slightly more general result is given in Hurwicz and Uzawa (1971) who provide the following sufficient conditions for a solution to the integrability problem to exist (Theorem 2, p. 124): (i) the Slutsky matrix (with ij^{th} element $s_{ij} = (\partial x_i/\partial p_j + x_j(\partial x_i/\partial m)$ is symmetric and negative semi-definite, (ii) the functions $\mathbf{x}(\mathbf{p}, m)$ satisfy the adding-up condition $\mathbf{px} = m$, (iii) $\mathbf{x}(\mathbf{p}, m)$ is differentiable with partial derivatives appropriately bounded (their condition E, i.e. continuity of the partial derivatives, would suffice). Broadly speaking then (subject to adding-up and regularity conditions such as (iii)) the standard Slutsky properties of symmetry and negative definiteness are necessary and sufficient for the existence of a utility function (the necessity part has, of course, been known since at least 1915 and the original Slutsky article). The method of proof of the existence theorem provided by Hurwicz and Uzawa (1971) also shows how one may attempt to construct the utility function, as we shall see in our discussion of the third criterion.

In the case of the simple model with two goods, i.e. consumption, c, and labour, l, with demand/supply functions $c(w, m)$ and $l(w, m)$ and budget constraint

$$c \leqslant wl + m, \tag{1}$$

where w is the wage, all that is required in addition to a regularity condition such as (iii) (which we shall largely ignore) is that the wage response of the compensated supply of labour be non-negative, i.e.

$$\frac{\partial l}{\partial w} - \frac{l \partial l}{\partial m} \geqslant 0. \tag{2}$$

Thus given a labour supply function consistency with utility maximisation may be checked using (2) very easily.

2. *Convenience in estimation*

Our discussion of this issue will be superficial since a vast literature exists on the estimation of the labour supply function (see Killingsworth, 1983, for a substantial survey). The aspects we shall briefly indicate are simply (i) the possibility of estimation by linear regression, (ii) the ease with which differences across households may be incorporated, (iii) the plausibility of the stochastic specifications which arise conveniently in association with the functional form.

3. *Availability of tractable direct utility and indirect utility (or cost) functions*

The proof of the existence of a solution to the integrability problem by Hurwicz and Uzawa (1971) shows how one can attempt to construct an explicit utility function. If $m(\mathbf{p}, u)$ is the cost or expenditure function associated with the utility function, then we know that

$$\frac{\partial m}{\partial p_i} = x_i(\mathbf{p}, m), \tag{3}$$

where m on the rhs of (3) is evaluated at $(\mathbf{p}, u)$. Thus (3) is, for constant u, a system of partial differential equations for the function m: the constant of integration characterises the utility level and it (or some transform of it) may be used as a utility index. The crux of the Hurwicz–Uzawa proof is the existence theorem for a solution to the set of equations (3).

The method of finding utility functions corresponding to demand/supply functions by integrating (3) has been used in the context of labour supply by Deaton and Muellbauer (1981) and (essentially) in a series of papers, Burtless and Hausman (1978); Hausman (1980, 1981*a* and *b*). Hausman generally uses the version of (3) constructed using Roy's identity which says that, where $v(\mathbf{p}, m)$ is the indirect utility function,

$$x_i(\mathbf{p}, m) = -\frac{\partial v}{\partial p_i} \bigg/ \frac{\partial v}{\partial m}. \tag{4}$$

The integration of (4) to find m as a function of $\mathbf{p}$ at constant v gives indifference curves in $(\mathbf{p}, m)$ space which are the contours of the indirect utility function.

When we are studying labour supply (3) becomes, on treating the supply of labour as a negative demand,

$$\frac{\partial m}{\partial w} = -l(w, m), \tag{5}$$

where m is the 'unearned' or 'lump-sum' income required to reach utility u when the wage is w. Following Hausman and Deaton and Muellbauer, we present in the next section straightforward solutions to the integration of (5) for a number of different functional forms for $l(w, m)$, particularly the linear, log-linear and quadratic. For many functional forms for $l(w, m)$ the ordinary differential equation (5) could be integrated numerically. However it is convenient to have solutions in closed form and in the next sub-section we shall be presenting various simple examples where closed form solutions are available.

If we can specify a solution to the integration of (3) and thus find a utility function corresponding to the demand function, then a simple change of variable allows us to generate analogously another corresponding pair of demand and utility functions. Hence there are immediate counterparts to the functions such as the linear which may be integrated easily. The change of variable simply requires writing

$$\omega_i = \frac{p_i x_i}{m}, \qquad \text{then}$$

$$x_i \to \omega_i; \qquad m \to \log m; \qquad p_i \to \log p_i. \tag{6}$$

Then if we have demand functions in the form (7), where $\log \mathbf{p}$ is the vector with i^{th} component $\log p_i$,

$$\omega_i = g_i(\log \mathbf{p}, \log m), \tag{7}$$

we also have, corresponding to (3),

$$\frac{\partial \log m}{\partial \log p_i} = g_i(\log \mathbf{p}, \log m). \tag{8}$$

Since, if $m(\mathbf{p}, u)$ is the cost function as before, $(\partial \log m)/(\partial \log p_i)$ is ω_i. The problem of finding an integral in (8) is precisely the same as that of finding an integral for (3). Hence if we have provided the cost function for a given demand function we have also found the logarithm of the cost function for the demand system with the share of expenditure on each good having the same functional form with arguments the logarithms of prices and income. The corresponding indirect utility function for (7) is obtained from the transformations $\mathbf{p} \to \log \mathbf{p}$ and $m \to \log m$, in the $v(\mathbf{p}, m)$ associated with (3).

In considering the notion of share of expenditure when applied to labour supply we can work either with $-wl/m$ or $[w(T-l)]/(m + wT)$ on the lhs of (7) where in the latter case T is total time, $T-l$ is 'leisure' and $m + wT$ is 'full income'. A problem with this second approach is that it is not easy to provide satisfactory definitions or measures of T for applied work. A difficulty with the former approach arises if m is zero or negative, a common occurrence in cross-sections. In either case the shares of consumption and labour/leisure add to one, although in the former case the individual shares do not lie between zero and one.

Once the cost function $m(\mathbf{p}, u)$ has been derived by integration, the indirect utility function is found by inversion, i.e. $v(\mathbf{p}, m)$ satisfies $m(\mathbf{p}, v) = m$. The direct utility function may be found in the standard way by using the result that for a given $v(\mathbf{p}, m)$, $u(\mathbf{x})$ is the solution to

$$\underset{\mathbf{p}, m}{\text{minimise}} \{v(\mathbf{p}, m) \colon \mathbf{px} = m\}, \tag{9}$$

where, without loss of generality, we can put $m = 1$. Equivalently, one can substitute, for $\mathbf{p}/m$ in $v(\mathbf{p}/m, 1)$ using the inverse demand functions which express $\mathbf{p}/m$ as functions of the quantities, $\mathbf{x}$.

Given that the routes between supply function, indirect utility (or cost) function and direct utility function are well established through (5) and (9), one can choose to begin with whichever of the functions happens to be most convenient. One would like tractable forms for all three functions but this will be possible only for certain examples. And convenient forms for the *inverse* supply function $w(c, l)$ are also desirable as we shall see in our discussion of criteria (5) and (6) (the issue is also relevant for estimation–see e.g. Heckman, 1974).

4. *Facility of use in applied problems*

The applications of an estimated labour supply function and associated utility function may involve, for example, on the positive side, prediction of future labour supplies and the response to wage, price or tax changes and, on the normative, the evaluation of tax changes through their effects on household welfare and government revenue. At various points in these analyses it may be important to have tractable forms for all three labour supply, indirect utility (or cost) and direct utility functions. The labour supply function should be tractable for calculating responses with a linear budget constraint and one may need the indirect utility function for estimation with non-linear constraints (see e.g. Burtless and Hausman, 1978). And in a number of contexts, for example where rationing is important or there are errors in choice or the budget constraint is not piece-wise linear, it may be convenient to have explicit forms for the direct utility function. The relative importance of the tractability of the three functions will depend on the context. Only in rare examples will all three functions be easy to deal with and therefore we may wish to choose to start with the direct utility function if a tractable form for this is particularly important or, for example, the labour supply function if facility in this direction is especially convenient. The choice will depend on the circumstances; thus one may use different representations of utility for different applications.

The question of facility of use in applied problems would therefore be largely decided by the application of the second and third criteria of Table 9.1. Additionally, however, one wants to introduce a further aspect: the transparency of the important parameters. This is important for a number of reasons.

Firstly, one may want to analyse the sensitivity of predictions or judgments to assumptions concerning, or estimates of, supply functions. Thus one may wish to ask, for example, how the predictions would be affected if the compensated or uncompensated wage elasticity of supply of labour were higher. If that is an important question then one wants to choose a functional form where that question can be conveniently posed with clear assumptions about what is being held constant. If one uses a CES utility function then it is straightforward to examine sensitivity of results to the elasticity of substitution (see e.g. Stern, 1976) but this does not directly answer a question about e.g. sensitivity to the compensated wage elasticity holding the income elasticity constant. Secondly, it will often be important to compare estimates from one source with those from another which may use different methods, data or forms, Thus one may want to ask how the income elasticity calculated in paper A compares with that in paper B and it would then be desirable that the forms chosen and the presentation of results allow comparisons of this kind. Thirdly, intuitive explanations of how results come about often work in terms of simple parameters such as wage or income elasticities or elasticities of substitution and thus in order to understand one's own conclusions it is important to be able to extract these simple notions fairly easily.

5. *Facility for use in theoretical problems*

The particular application we have in mind is that of optimum income taxation with non-linear taxes–see Mirrlees, 1971. In this case one works mostly with a direct utility function rather than the indirect forms (although these are useful for linear tax models, e.g. Stern, 1976). Thus tractability of the direct utility function is desirable. There are also aspects of the utility or labour supply functions which make the formulation of the problem and computations of solutions convenient.

First, we require that $u(c, l)$ should be such that the associated consumption demand and labour supply functions $c(w, m)$ and $l(w, m)$ (which arise from the maximisation of $u()$ subject to the linear constraint $c = wl + m$) have the property that

$$\frac{\partial c}{\partial w} > 0. \tag{10}$$

This ensures that pre-tax earnings will be an increasing function of the wage for any relevant income tax function (see Mirrlees, 1971, Theorem 1). This last feature is important to most standard formulations of the income tax problem in that it implies that higher wage individuals have higher income and thus allocate themselves to incomes in the same order as their wage rates. The condition (10) is simply that consumption is not a Giffen good (w is the inverse of the price of consumption) and thus a sufficient condition is that consumption be a normal good ($\partial c/\partial m > 0$). Given a labour supply function $l(w, m)$, and thus also consumption demand $c(w, m)$, using $c = wl + m$, the signs of the derivatives of c are readily checked, as of course are the derivatives of l. It seems reasonable to expect $\partial c/\partial w$ and $\partial c/\partial m$ to be positive and $\partial l/\partial m$ to be negative with $\partial l/\partial w$ ambiguous. Note that $\partial c/\partial m = (w\partial l/\partial m) + 1$ so there is a potential conflict between normality of consumption ($\partial c/\partial m > 0$) and of leisure ($\partial l/\partial m < 0$). One would not necessarily expect leisure to be a normal good at very low levels of labour supply (see the next criterion).

A second and stronger condition which makes formulae much simpler and calculations of the optimum easier is the existence of a representation of utility which is additively separable, i.e. a utility function $u(c, l)$ with $\partial^2 u/\partial c\partial l = 0$. The condition is sufficient for (10) (with strict quasi-concavity of $u()$). It should be emphasised that whilst additive separability eases calculations considerably, it is not required for the usual formulation of the problem and the derivation of first-order conditions.

A necessary and sufficient condition for the existence of an additively separable representation of the preferences corresponding to a labour supply function is

$$\frac{\partial^2 \log w}{\partial c\partial l} = 0, \tag{11}$$

where $w(c, l)$ is the inverse demand function for the demand/supply functions $c(w, m)$ and $l(w, m)$ and we suppose that the demand/supply functions are invertible. The characterisation of necessary and sufficient conditions in the two-good model was provided by Samuelson (1947, pp. 174–77).

The inverse demand function $w(c, l)$ may be derived as follows from $l(w, m)$. We substitute for m from the budget constraint $c = wl + m$ and change the variable from w to log w to have

$$F(\log w, l, c) = 0. \tag{12}$$

If this may be inverted easily we have

$$\log w = \phi(l, c) \tag{13}$$

and condition (11) may readily be checked. If not then we use

$$\frac{F_1}{F_2F_3}\frac{\partial^2 \log w}{\partial c\partial l} = \frac{F_{13}}{F_3F_1} + \frac{F_{12}}{F_1F_2} - \frac{F_{23}}{F_2F_3} - \frac{F_{11}}{F_1^2}, \tag{14}$$

where F_i is the partial derivative of F with respect to the i^{th} argument and F_{ij} the partial derivative of F_i with respect to the j^{th} argument. It is straightforward to evaluate the rhs of (14) and thus to check condition (11).

Notice that the condition (11) is on the *second* derivative of the inverse demand function. The usual condition for additivity, that the Slutsky term

s_{ij} be proportional to the product of the income derivatives of goods i and j, does not impose a constraint here. For a given preference structure in the two-good model any one indifference curve may be represented using an additively separable form (then additivity imposes strong relations between this curve and the others). With three or more goods marginal rates of substitution between any pair must be independent of a third good but this is not relevant in the two-good case. In the two-good case, the reservations about the empirical consequences of additive separability which arises, for example, in general demand analysis (see e.g. Deaton, 1974), do not appear to be so severe. One may regard, perhaps, additive separability in the labour supply context as possessing virtues for the income tax problem which are not wholly offset by undesirable consequences for the pattern of supply responses.

A major influence on calculated optimum income tax rates is exerted by the elasticity of substitution between leisure and goods (see Stern, 1976) and one would not want to choose functional forms which imposed tight constraints on this elasticity. In problems where the optimum choice of both commodity and income taxes are made then separability between labour and goods plays an important role. Under some rather strong and implausible conditions (identical preferences across households and differences arising only in the wage rate) then weak separability between leisure and goods implies that optimum commodity taxes are zero and all revenue is raised through the optimum non-linear income tax (see Atkinson and Stiglitz, 1980, Chapter 14). In the two-good (c, l) model considered here, weak separability is no restriction (additive separability *is*, of course, as we have just been discussing) so that this particular issue does not arise. The linear expenditure system for goods and leisure (LES) implies zero optimum commodity taxes (under the same conditions) even where we are restricted to the (optimum) linear income tax. We shall be discussing the LES further below (see e.g. criterion (7)) but note that linearity for expenditure on leisure as a function of w and m does not require the LES to hold for all consumption goods (see Muellbauer, 1981).

6. *Behaviour for low levels of labour study*

The forms of the utility function and labour supply function at low levels of hours are important both in estimation and in policy models. Where participation is an issue in estimation then one essentially compares utility when not working with utility when working for any individual, when calculating the probability that the individual will work. Thus the form of the function $u(c, l)$ and its partial derivatives around $l = 0$ may have considerable consequences for parameter estimates. Thus incorrect specification of the functions in this region may produce substantial biases in predicted responses at much higher levels of labour supply.

Secondly, in the theoretical models of optimum income taxation the numbers not working for any given income-tax function will be determined by the number of people with shadow wage (marginal rate of substitution of consumption for leisure) above their marginal net wage at zero labour supply. And the variation of this number with the tax function may have an important influence on which schedule is selected as optimum. Note that if the marginal disutility of labour is negative there will in general be no unemployment in these models.

Thirdly, one may ask about feasibility—what is possible for workers at low hours and low wages? This is the notion of the consumption set in standard economic theory and was discussed at length in Bliss and Stern (1978). A fourth consideration concerns disincentive effects and the poverty trap at low earnings. Low earnings are, one supposes, mainly a consequence of low wages but if they are in part a result of low hours (and the two will be connected through the supply curve) then we should pay special attention to functional forms at low levels of hours.

We shall not go into these issues in great detail but emphasise the importance of looking at the limits of $u(c, l)$ and $w(c, l)$ as l tends to zero for various values of c where $w(c, l)$ is the inverse supply function. Notice that this consideration and that in the previous section indicate that tractability of the inverse supply function is also a desirable attribute.

The investigation of criterion (6) involves then looking at $\lim_{l \to 0} u(c,l)$ and $\lim_{l \to 0} w(c, l)$ and discussing the plausibility of these limits. We suggest that it is an advantage of a utility function if it does not automatically rule out for low levels of labour supply two possibilities which might appear less attractive features at higher levels of labour supply. The first is that the marginal disutility of labour at zero labour supply may be negative (i.e. a little labour at constant consumption is utility increasing) and the second is that, where the marginal disutility of labour is positive at zero labour supply, then leisure may be an inferior good. In terms of the labour supply function the former condition would imply that it should not intersect the positive w-axis and the second that if it does intersect the positive axis then at this point $\partial l/\partial m > 0$. The second condition can be expressed in terms of the inverse supply function $w(c, l)$ by $\partial w/\partial c < 0$ at $l = 0$. To help avoid inferiority at higher levels of m one may wish to add the condition $\partial^2 l/\partial m^2 < 0$ at $l = 0$ (more yachts may make work less attractive).

If the supply curve does intersect the positive l-axis then there will be non-negative combinations (c, l) which would not be chosen for any positive wage since the marginal disutility of labour is negative (the virtual wage would have to be negative). The boundaries of the region are $c = 0$, $l = 0$ and $l = l(0, c)$ where l is the labour supply function.

Note that if the labour supply curve does intersect the positive w-axis then $\partial l/\partial w \geqslant 0$ at this point from the Slutsky condition at $l = 0$. This implies that if l is differentiable there is *at most one* intersection with this axis.

In most samples one observes a very small proportion (well below 1%) of the working population working less than, say, 16 hours per week (see e.g. Atkinson, Micklewright and Stern, 1982). The relative absence of people working a small number of hours might be taken as evidence in favour of the view that there is low or negative disutility of work at low hours. It is, at least, consistent with that view. This division of the population between those not working and those working more than 16 hours could then be explained, together with low disutility at low hours, in terms of fixed costs of working for the employee (see e.g. Hausman, 1980), fixed costs of employment for the employer, ineffective demand, and so on. Whilst the absence of people working very low hours is consistent with low or negative marginal disutility of labour, it also implies that the precise form of the utility function over this range may not be very important. What is important is that we allow the *possibility* of low or negative marginal disutility since it could have substantial effects on estimates of fixed costs of work and on labour supply estimation where the participation decision is endogenous.

7. *Aggregation*

Many data on labour supply come in aggregated form: for example, averages or totals for a region, group or year. Thus a reasonable question to ask of a labour supply function is whether it may be aggreagated across individuals. Specifically, individuals differ in their wage w and lump-sum income m, and one may wish to write aggregate or average labour supply as a function of the average wage and lump-sum income. Notice that the possibility of aggregation is a significant advantage only if one works with aggregate data. If the micro-data are available then there is no special reason to work with a function which can be aggregated since aggregates can be constructed directly by adding across households.

The standard results on aggregation (Gorman, 1953; Theil, 1954) tell us that necessary and sufficient conditions involve the linear function, i.e.

$$l^h = \alpha w^h + \beta m^h + \gamma^h, \tag{15}$$

where the superscript h for a variable denotes its value for household h ($h = 1, 2, \ldots, H$). Then

$$\bar{l} = \alpha \bar{w} + \beta \bar{m} + \bar{\gamma}, \tag{16}$$

where for a variable z, $\bar{z} = (1/H)\Sigma_h z^h$ and there are H households. Notice that α and β are common across households but γ need not be. Thus if characteristics which vary across households are introduced and we wish to preserve aggregation then they must be additive. Where the averages which enter the aggregate equation, such as (16), are arithmetic averages, then we call the aggregation *linear*.

Non-linearities in variables may be introduced provided we retain linearity in the parameters and the average is taken appropriately. Thus, for example, the log-linear form

$$\log l^h = \alpha \log w^h + \beta \log m^h + \gamma^h \tag{17}$$

can be aggregated to

$$\log \tilde{l} = \alpha \log \tilde{w} + \beta \log \tilde{m} + \bar{\gamma}, \tag{18}$$

where $\tilde{l}$ is the geometric mean (satisfying $\log \tilde{l} = (1/H)\Sigma_h \log l^h$). Similarly, one could work with linearity in z^ϵ and take harmonic means and so on. The problem in using (17) or similar forms is that we usually have in the data the ordinary arithmetic average, $\bar{z}$, of variables.

If we ask for linear aggregation of *each equation* of the demand system then (15) will not do, a point which has been emphasised by Muellbauer (1981). The argument can be demonstrated in our simple two-good (c and l) system as follows. If (15) holds then the budget constraint $c^h = w^h l^h + m^h$ implies

$$c^h = \alpha(w^h)^2 + \beta w^h m^h + \gamma^h w^h + m^h, \tag{19}$$

which does not permit aggregation. The analogous argument follows with many goods: the basic point is that the budget constraint represents a linear sum of expenditures, not quantities, and if we are to have aggregation, and thus linearity, in each equation whilst preserving the budget constraint, then the linearity must be in terms of expenditures. Thus

$$w^h l^h = \alpha w^h + \beta m^h + \gamma^h \tag{20}$$

permits aggregation of both labour supply and commodity demand equations. Notice that the lhs of the aggregated equation (20) involves average earnings $(1/H)\Sigma_h w^h l^h$. This is equal to the product of the average wage $\bar{w}$ and an hours *index* $\hat{l}$ where $\hat{l} \equiv \Sigma_h (w^h/\Sigma_h w^h) l^h$ or a *weighted* average of hours where the weights are $w^h/\Sigma_h w^h$ (see Muellbauer, 1980). Similarly average earnings is the product of average hours and a wage index $\hat{w} \equiv \Sigma_h (l^h/\Sigma_h l^h) w^h$ which is the weighted average of wages using hours worked as weights.

It should be clear that linearity is required only in those variables which vary across households. Thus in a labour supply and commodity demand system with several consumption goods, prices which are common to all households may appear in a non-linear way and still preserve linear aggregation with expenditure on each commodity a linear function of w and m. Thus one is not confined to the linear expenditure system (where expenditures are linear functions of prices as well as w and m) for commodities and the coefficients in (20) may depend on the prices p. The coefficient γ must be homogeneous degree one in p and α and β homogeneous

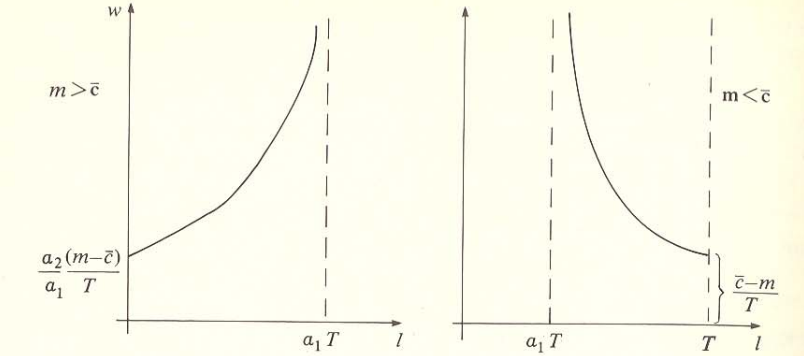


Fig. 9.1. The linear expenditure system.

degree zero. Where we ask for consistency with utility maximisation then α, β, γ^h must be such that they can be derived from an appropriate cost function (see Muellbauer, 1981, for further details).

It should be noted, however, that as far as labour supply and our two-good (c, l) model is concerned (20) is simply the linear expenditure system. If we write the two-good LES utility function in standard form, with $\alpha_1 + \alpha_2 = 1$,

$$u(c, l) = \alpha_1 \log (c - \bar{c}) + \alpha_2 \log (T - l), \qquad (21)$$

where $\bar{c}$ is minimum consumption and T maximum labour supply, then maximisation subject to $c = wl + m$ gives, for an interior solution,

$$wl = \alpha_1 wT - \alpha_2 (m - \bar{c}). \qquad (22)$$

Comparing (20) and (22) α corresponds to $\alpha_1 T$, α_2 to $-\beta$ and γ^h to $\alpha_2 \bar{c}$. Assuming α_2 is positive the labour supply curve (w as a function of l for given m) is entirely forward sloping if $m > \bar{c}$ and entirely backward sloping if $m < \bar{c}$. It is a highly restrictive labour supply curve and is illustrated in Figure 9.1.

But the LES does not necessarily force the backward-bending supply curve as Muellbauer, 1981, p. 27 suggests. Further, as regards labour supply as a function of wage, the general system that permits labour supply/commodity demand aggregation is *no more* flexible than the LES (notwithstanding Muellbauer's remarks to the contrary on p. 27, 1981), although when there are several goods it is much less restrictive than the LES in terms of cross-elasticities between labour and other goods.

The issue of linear aggregation is therefore reasonably straightforward as embodied in (15) and (20). As far as the dependence of labour supply on the wage is concerned we require either the linear supply function or the LES. If we also require linear aggregation of consumption as a function of the wage and lump-sum income, then we are restricted to the LES for labour supply.

It should be stressed that with *several* consumption goods the LES for the whole system is *not* implied.

8. *Flexibility in the relation between labour supply and the wage*

Studies of cross-section data for the US on the response of hours worked by adult males to wages have often found that labour supply increases with the wage for low wages and then decreases at higher wages (see e.g. Hall, 1973). On the other hand, some cross-section studies using data for the UK have found, for adult males, non-monotonicity in the opposite direction (see e.g. Brown, Levin and Ulph, 1976; and Atkinson Stern and Gomulka, 1980) where labour supply first decreases with the wage and then increases. Labour supply studies for married women (see e.g. Killingsworth, 1973, Chapters 3 and 4) have found responses to be rather more elastic for women than for men. The findings of changes of curvature or non-monotonicity are, of course, related to the methods used and some of these may not be wholly satisfactory. But the range of possibilities which has been found, and indeed ordinary caution, should suggest to us that we should be wary of working with functional forms which automatically impose given signs on responses, which rule out changes in the sign of slopes of labour supply response, or which tightly constrain changes in curvature.

A related but different question concerns the number of parameters to be estimated. A minimum would appear to be three since we have to consider responsiveness of labour supply to the wage, and to lump-sum income together with its general level (or constant term). A form with just two parameters such as the CES will clearly involve restrictions amongst these features which may be unsatisfactory and should at least be tested. Diewert (1974), in particular, has emphasised the importance of avoiding this type of restriction.

It is perhaps unfortunate that the recent advances in statistical techniques used in estimating labour supply responses have often gone hand in hand with rather simple and inflexible functional forms, although one can appreciate the reluctance to add any complication. But inflexibility in the functional forms used may produce very misleading predictions and policies. If, say, the labour supply curve is actually forward sloping for only some of its range and we use a form such as the linear which (with positive wage coefficient) forces it to be upward sloping throughout, then we might find ourselves predicting that a cut in income tax would lose little or no revenue when in fact it would lose a great deal. Our eighth criterion, and the one which originally motivated this paper, is that the form should allow a variety of possible responses and, in particular, the possibility of substantial changes in the slope of the response of hours to wages.

9.3 Functional forms

Our purpose in this section is to show how some commonly used functional forms stand in relation to the criteria we have suggested in the preceding section. Whilst we shall be discussing several we shall not be exhaustive and we hope that, given the methods described above and the examples given below, it will be reasonably straightforward to carry through the exercise for functions which are not given here. We shall also be suggesting a class of utility functions and labour supply functions which arise from the methods described but which do not seem to have received much attention.

As we pointed out in Section 9.2, and in particular in our discussion of criterion (3), one can start from the supply function and integrate to find the cost function and thus indirect and direct utility functions, or start with one of the representations of the utility function and derive the labour supply function. Our first group of examples begins with the labour supply function and the second group the utility function. In the first group we have the linear, quadratic and log-linear together with the corresponding versions in shares for the linear and quadratic cases, as we indicated in Section 9.2 (criterion (3)). In the second we have the linear expenditure system (LES), the constant elasticity of substitution (CES) together with an extension, examples of so-called 'flexible forms' for the utility function and, finally, the LES extended to the case where consumption involves time (see Atkinson, Stern and Gomulka, 1980). The notion of a 'flexible form' can be expressed through the supply function as well as the utility function so that we can view, for example, the quadratic as a 'flexible form' for labour supply.

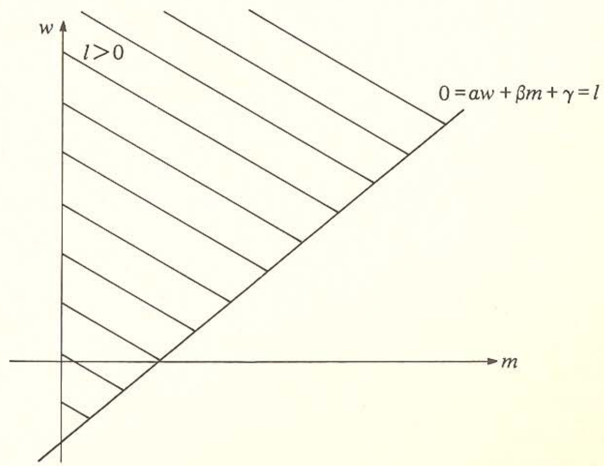


Fig. 9.2. Non-negativity of l and the Slutsky condition for linear labour supply function.

We shall present the relation between the functional forms and the criteria in terms of a table for each of the forms considered so that the results are available in a systematic and easily accessible form. A brief commentary will be provided on each of the tables which appear at the end of the paper. In Table 9.2 we summarize the functional forms we shall consider.

The linear supply function: commentary on Table 9.3

The Slutsky conditions holds for $l \geqslant \alpha/\beta$ if β is negative, so if $\alpha \geqslant 0$ then it holds wherever $l \geqslant 0$. The region of (w, m) space with positive l is shown in Figure 9.2 which has been drawn for $\alpha > 0$, $\beta < 0$, $\gamma > 0$. The expenditure function can be found from integrating

$$\frac{\partial m}{\partial w} + \beta m = -\alpha w - \gamma, \tag{23}$$

which simply involves the integrating factor $e^{\beta w}$. All the important functions are available explicitly in a tractable form. If the curve intersects the w-axis, then, at the axis $\alpha > 0$ for Slutsky. An unsatisfactory feature is that it is impossible for leisure to be inferior for low l and normal for high l. If leisure is normal then consumption is inferior for high w. An unappealing aspect is the inflexible response of l as a function of w.

The quadratic supply fraction: commentary on Table 9.4

The Slutsky condition involves the sign of a cubic in (w, m). The region in which it holds is not difficult to check for given values of the parameters but is complicated for the general case. The expenditure function can be found from integrating

$$\frac{\partial m}{\partial w} + (\beta + \nu w)m + \mu m^2 = -\alpha w - \lambda w^2 - \gamma. \tag{24}$$

For $\mu = 0$ this simply involves the integrating factor $e^{\beta w + (\nu/2)w^2}$ which yields expenditure and indirect utility functions which are straight-forward although the direct utility function is less tractable. The case $\mu = \nu = 0$ is treated in Hausman (1981) but that for $\nu \neq 0$ is not, and we see that, using the standard function 'erf', this case too is tractable.

If $\mu = \nu = 0$ then it is impossible for leisure to be inferior for low l and normal for high l; however, for $\mu = 0$, $\nu < 0$ and $\beta > 0$ we can have $\partial l/\partial m > 0$ for low w and < 0 for high w. The overall flexibility of response permitted by the form appears satisfactory.

The extra flexibility from introducing the terms in w^2 and wm seems justified in the sense that an important generalisation is introduced at little

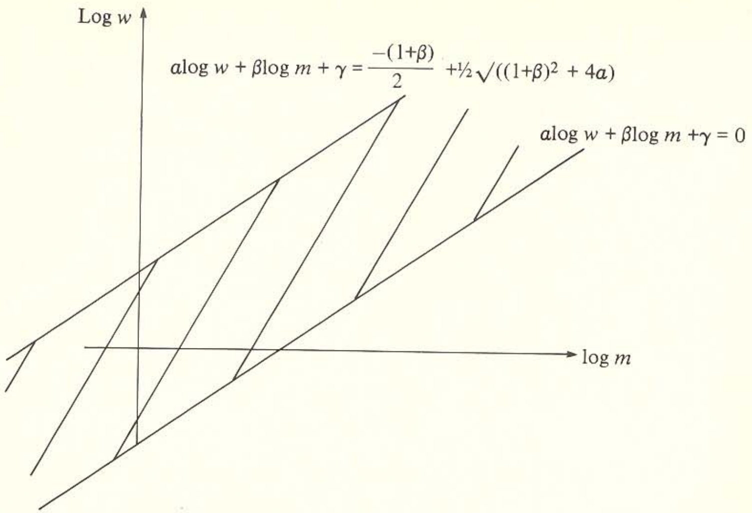


Fig. 9.3. Positivity of labour supply and the Slutsky condition for 'share linear in logarithms' (Table 9.7). Case $\alpha > 0, \beta < 0$.

extra cost. The extra terms can be introduced sequentially and as the forms $\mu = 0$ and $\nu = 0$ are nested in the more general case one can employ standard tests.

The log-linear supply function: commentary on Table 9.5

If $\alpha > 0$ and $\beta < 0$ then the Slutsky condition holds and if $\alpha < 0$ and $\beta > 0$ it is violated. For the other cases it will hold over a subset of (w-m) space (e.g. $w^{\alpha+1} m^{\beta+1} < \alpha/\beta k$ for $\beta > 0$). The expenditure function can be found from integrating

$$\frac{\partial m}{\partial w} = -k w^{\alpha} m^{\beta}, \tag{25}$$

which is separable in the variables w and m. The expenditure and indirect utility functions are straightforward although the direct utility function is less tractable. It is impossible for leisure to be inferior for low l and normal for high l. If leisure is normal then consumption is inferior for high w or low m. It is not possible for l to be zero making the function inappropriate for studies of participation. The response of l to w is inflexible with a constant elasticity and the exclusion of non-monotonicity.

The semi-log supply function: commentary on Table 9.6

The Slutsky condition is satisfied for $\alpha > 0$ and $\beta < 0$ and is violated for $\alpha < 0$ and $\beta > 0$. For the other cases it will hold over a subset of (w-m) space. The expenditure function can be found from integrating

$$\frac{\partial m}{\partial w} + \beta m = -\alpha \log w - \gamma, \tag{26}$$

which simply involves the integrating factor $e^{\beta w}$ and the integration by parts of $e^{\beta w} \cdot \log w$. The expenditure and indirect utility functions are straightforward although the direct utility function is less tractable. It is impossible for leisure to be inferior for low l and normal for high l. If leisure is normal then consumption is inferior for high w. The response of l to w is inflexible.

Shares linear in logarithms: commentary on Table 9.7

It is straightforward to check that the Slutsky condition requires

$$\alpha - (1+\beta)\frac{wl}{m} - \left(\frac{wl}{m}\right)^2 \geqslant 0,$$

which gives the condition specified in Table 9.7. We also require $wl/m > 0$, hence the Slutsky condition together with positivity of l becomes (see Figure 9.3)

$$\max(0, -(1+\beta)/2 - \tfrac{1}{2}\sqrt{((1+\beta)^2 + 4\alpha)}) \leqslant \alpha \log w + \beta \log m + \gamma$$
$$\leqslant \frac{-(1+\beta)}{2} + \tfrac{1}{2}\sqrt{((1+\beta)^2 + 4\alpha)}. \tag{27}$$

The expenditure function can be found from integrating

$$\frac{\partial \log m}{\partial \log w} + \beta \log m = -\alpha \log w - \gamma, \tag{28}$$

which is the same form as (23) with the substitution $w \to \log w$ and $m \to \log m$. The expenditure and indirect utility functions are straightforward although the direct utility function is less tractable. If β is negative then it is possible for leisure to be inferior for low l and normal for high l. The shape of the labour supply curve as a function of w is sketched in Figure 9.4. For positive α non-monotonicity is possible although only in the direction shown. This functional form does not seem to have been greatly used in the literature, although it is essentially the AIDS form applied to labour supply (see Deaton and Muellbauer, 1980).

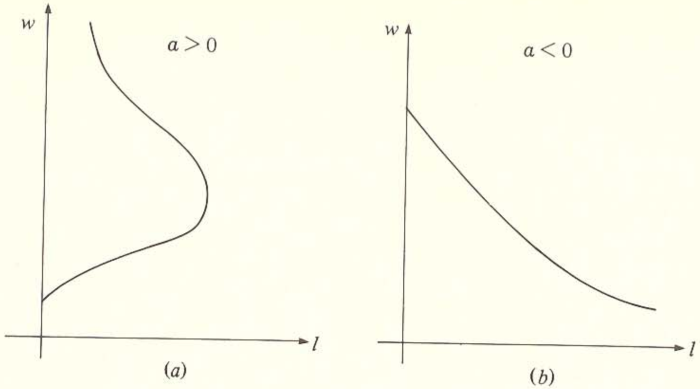


Fig. 9.4. Labour supply curve for 'share linear in logarithms' (Table 9.7).

Share quadratic in logarithm: commentary on Table 9.8

We concentrate on the case $\mu = \nu = 0$. The Slutsky condition involves putting α to $\alpha + 2\lambda$ log w in the condition for Table 9.7. The subset of (log w, log m) space for which it holds will no longer be as simple as Figure 9.3 but the boundaries will take a similar form with the two straight lines becoming curves. The expenditure and indirect utility functions are straightforward although the direct function is less tractable. If β is negative then it is possible for leisure to be inferior for low l and normal for high l. There are a number of possibilities for the shape of the labour supply curve as a function of w. For $\lambda > 0$ there are 0, 1 or 2 values of w such that $l \geqslant 0$ and for which $\partial l/\partial w = 0$, whereas for $\lambda < 0$ there is exactly one (the case $\lambda = 0$ is given in Figure 9.4). We sketch the four possibilities in Figure 9.5.

Linear expenditure system: commentary on Table 9.9

All the relevant functions are tractable and Slutsky is satisfied for the range of values of the parameters given in Table 9.9. It is impossible for leisure to be inferior for low l and normal for high l. The response of l to w is inflexible taking one of the forms sketched in Figure 9.1.

CES: commentary on Table 9.10

All the relevant functions are tractable and Slutsky is satisfied for the given values of the parameters. It is impossible for leisure to be inferior for low l and normal for high l. In sketching l as a function of w there are four cases to consider according as $m - \bar{c} \gtreqless 0$ and $\epsilon \gtreqless 1$. These are illustrated in Figure 9.6.

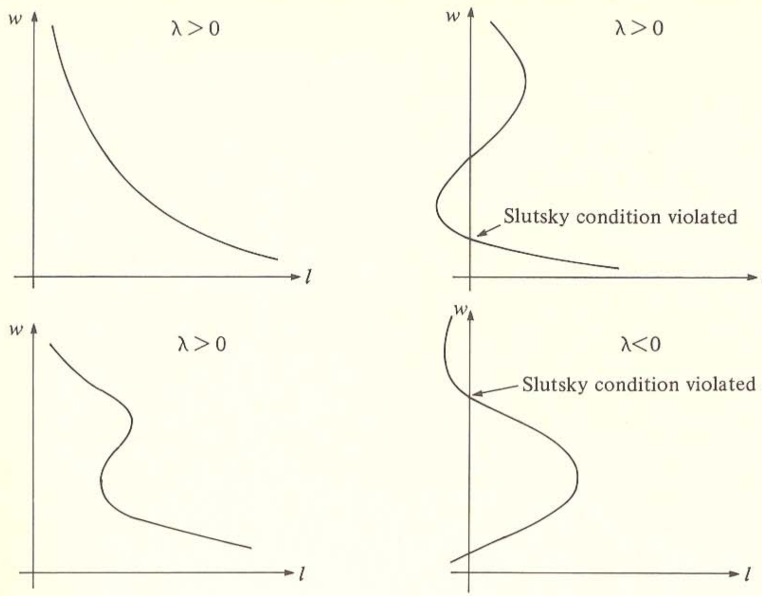


Fig. 9.5. Possible labour supply curves for share quadratic in logarithms (see Table 9.8).

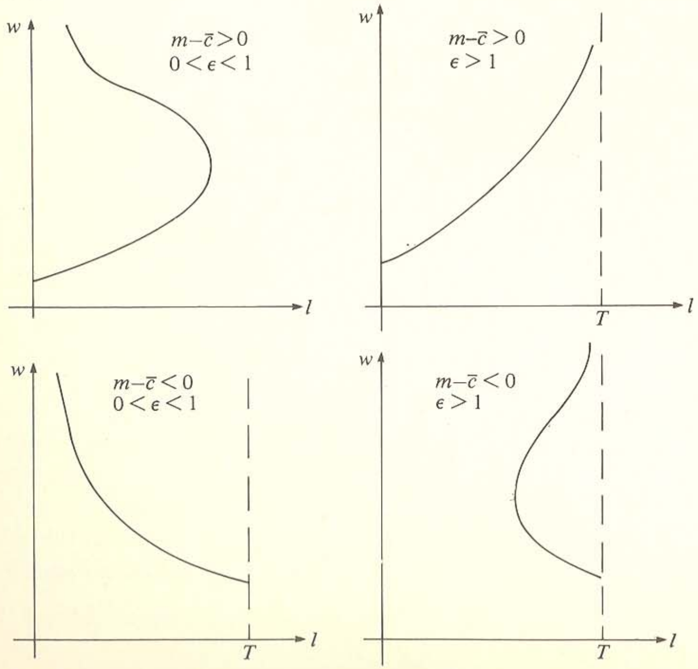


Fig. 9.6. Labour supply functions for the CES utility function (see Table 9.10).

Quadratic direct utility: commentary on Table 9.11

The Slutsky condition involves a simple inequality on w if $\alpha = 0$, otherwise it is messy. The indirect utility function is straightforward but the expenditure function is intractable. For negative α we can have $\partial l/\partial m$ negative for low w and positive for higher w although these do not necessarily correspond to high and low l respectively. The supply curve may be written

$$l = \frac{aw + b}{2(\alpha w^2 + \gamma w + \beta)} \quad \text{where} \quad \begin{aligned} a &= \epsilon + 2\alpha m \\ b &= \gamma m + \delta \end{aligned} \tag{29}$$

A number of cases can arise depending on the signs of a and b and whether w_1 and w_2, the roots of the quadratic $(\alpha w^2 + \gamma w + \beta) = 0$, are real. Four of these cases are illustrated in Figure 9.7. We suppose $\alpha < 0$.

Quadratic indirect utility: commentary on Table 9.12

Convexity and monotonicity of the indirect utility function hold under the conditions indicated in Table 9.12. The indirect utility function and labour supply functions are tractable but the direct is not and the expenditure function is only if $\alpha = 0$. In this case the labour supply function takes the same form as for the quadratic direct utility case. Leisure can be inferior for low w and normal for high w which corresponds to low l and high l if $\partial l/\partial w > 0$. The sign of $\partial l/\partial w$ is independent of w and $\partial l/\partial w$ increases with m.

Indirect translog: commentary on Table 9.13

Convexity and monotonicity of the indirect utility function hold under the conditions indicated in Table 9.13. The direct utility function and labour supply functions are intractable. The derivative of l with respect to m can change sign. The supply curve may be written

$$l = m \frac{2\beta \log w + a}{w(\gamma \log w + b)} \quad \begin{aligned} a &= \gamma \log m + \delta \\ b &= 2\alpha \log m + \epsilon \end{aligned} \tag{30}$$

Then $l = 0$ where $w = e^{-a/2\beta}$, and $l = \infty$ where $w = e^{-b/\gamma}$. As $w \to \infty(0)$, l has the sign of β/γ and tends to $0(\infty)$. The derivative $\partial l/\partial w$ vanishes at the roots of a quadratic in $\log w$. A number of cases arise depending on the signs of β and γ, whether $e^{-a/2\beta}$ is greater or less than $e^{-b/\gamma}$ and whether the quadratic has real or imaginary roots. Four of these cases are sketched in Figure 9.8.

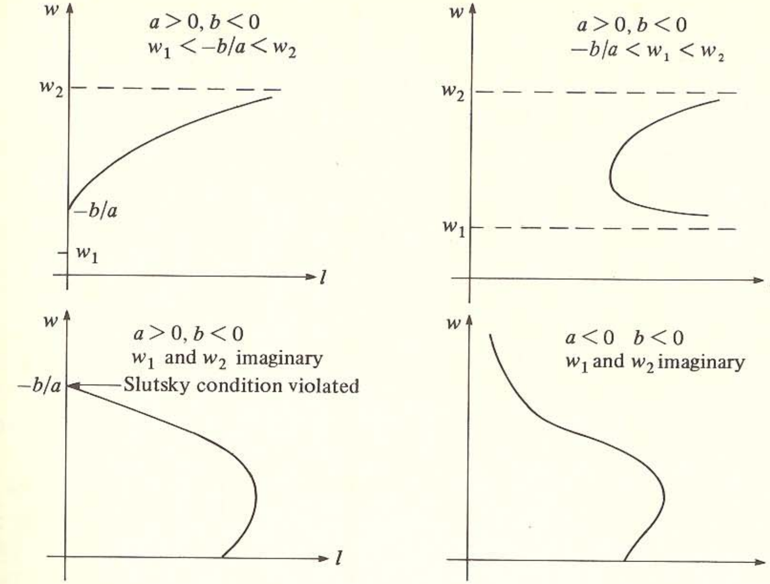


Fig. 9.7. Possible labour supply functions for the quadratic direct utility function (see Table 9.11).

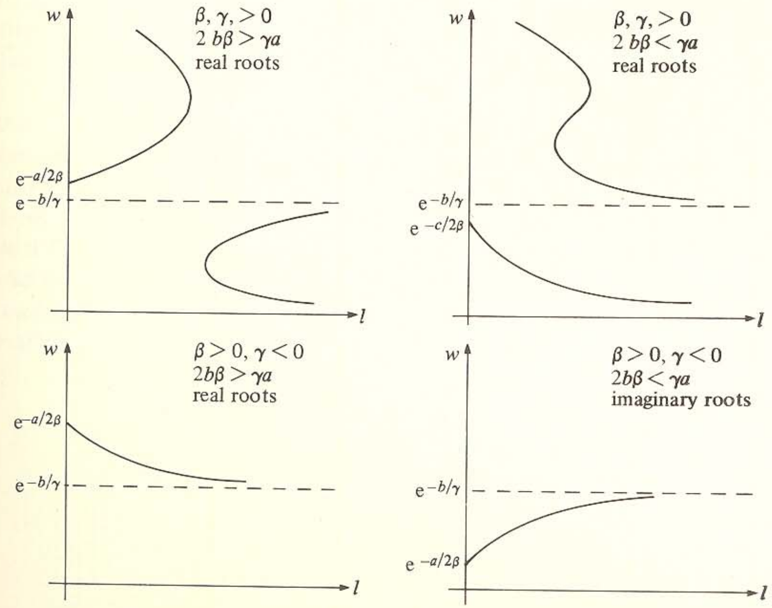


Fig. 9.8. Possible supply curves for the indirect translog utility function (see Table 9.13).

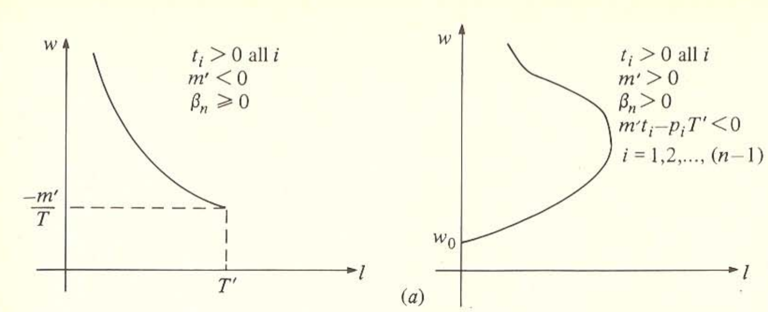
LES where consumption activity involves time: commentary on Table 9.14

In this model the consumption of activity i at level c_i requires the purchase of $p_i c_i$ worth of goods and uses $t_i c_i$ of time. Hence the effective price per unit of the activity for someone with wage w is $(p_i + wt_i)$. Thus the demand functions can be derived in the usual way by maximising utility as a funcion of $\mathbf{c}$ subject to $\Sigma_{i=1}^{n} (p_i + wt_i) c_i = m + wT$. Pure leisure (activity n) has $t_n = 1$ and $p_n = 0$.

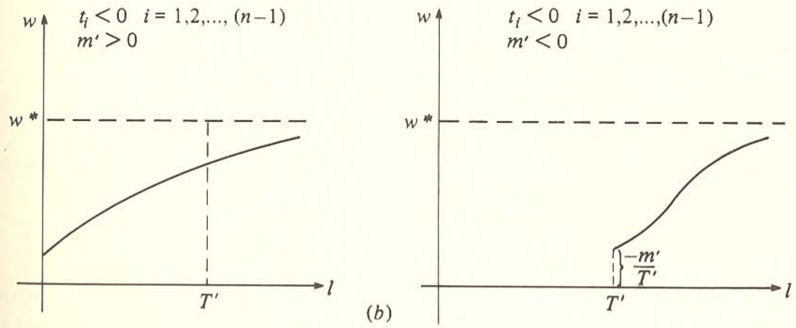
The standard Slutsky conditions are satisfied and all the relevant functions are tractable. We have $(\partial^2 l)/(\partial m \partial w) > 0$ so $\partial l/\partial m$ increases with w. Hence it decreases with l only if $\partial l/\partial w < 0$. It is possible for $\partial l/\partial w$ to change sign as w increases (more than once) and a number of shapes for the labour supply curve are possible. The four cases (*a*)–(*d*) given in Table 9.14 are sketched in Figure 9.9.

We can gain some intuition into the possible shapes as follows. If some t_i is negative then as w approaches $-p_i/t_i$ from below, the effective price of activity i approaches zero and we try to consume an infinite amount. The associated labour time can be achieved since the activity saves time. The model is invalid for $w > w^*$ where w^* is the minimum of $-p_j/t_j$ over those j for which $t_j < 0$. The demand function for activities gives 'expenditure' on activity i above the minimum level $(p_i + wt_i)(c_i - \gamma_i)$ as a constant fraction of 'super-numerary' income $(m' + wT')$. This last amount must be positive; hence w cannot fall below $-m'/T'$ and the model is valid only for $\max(0, -m'/T') \leqslant w < w^*$.

Activities which involve the saving of time become more attractive as the wage increases, since their price falls; extra consumption then releases more time for labour and makes for an upward-sloping curve–their contribution to $\partial l/\partial w$ is always positive. However, activities which require time in this sense compete with labour: if $p_i/m' < (wt_i/wT')$ so that the time cost is seen as high relative to money cost (expressed in terms of endowments) we have a negative contribution to $\partial l/\partial w$ and an influence in favour of a backward-sloping curve. The interaction of these influences can give rise to the different patterns shown in Figure 9.9.



Notes: (i) $\beta_n = 0$ then w_0 may be negative and the gradient may be negative throughout the positive orthant.
(ii) If some $t_i = 0$ and others > 0 then the limit of l as $w \to \infty$ is strictly positive.
(iii) Writing $m't_i - p_iT' < 0$ as $p_i/m' < wt_i/wT'$ we may interpret the condition as saying that the money element in the price (relative to net money endowment) is small relative to the time element (relative to net time endowment).



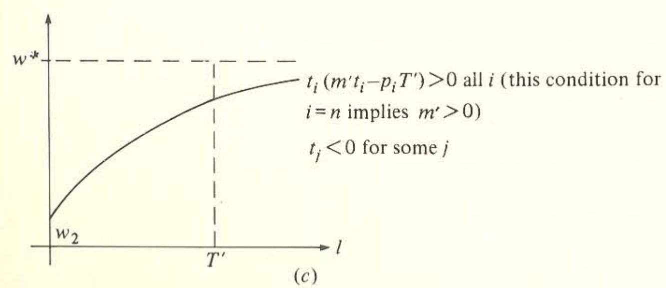
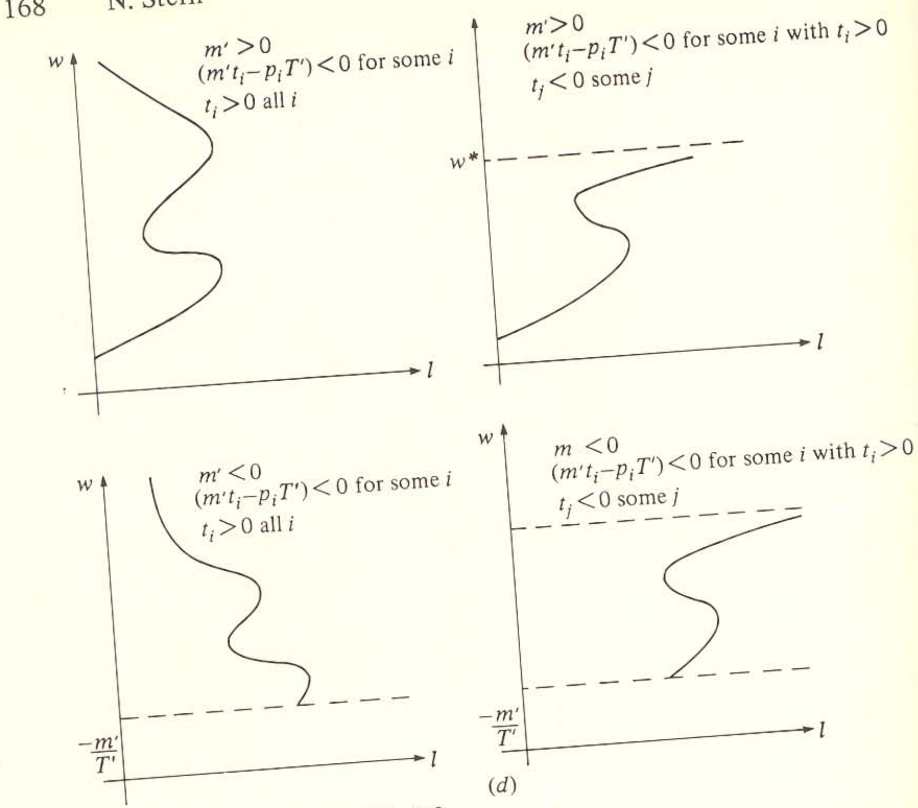


Fig. 9.9a, b, c. Labour supply curve for LES where consumption activity involves time (see Table 9.14).



See Notes (i), (ii) and (iii) for Fig. 9.9.

Fig. 9.9d. Labour supply curve for LES where consumption activity involves time (see Table 9.14).

9.4 An assessment of the functional forms

In the preceding section we examined the eleven different functional forms summarised in Table 9.2 to see how they fare against the criteria specified in Table 9.1. We shall now attempt to draw some lessons from this exercise.

The most general conclusion is that the choice of functional form for convenience in use and in estimation is likely to exert a considerable influence on prediction and the judgment of policies. This is familiar from optimum taxation problems where we know that the optimum commodity and income taxes can be not only very sensitive to parameter values but are also strongly influenced by the functional forms themselves (see e.g. Atkinson, 1977; Deaton, 1981). Similar problems arise in predicting the consequences of changes in taxes for behaviour and welfare where, again, strong bias in our policy conclusions and predictions may arise simply from choosing a functional form which may be convenient for estimation. Of special concern here is the forcing of inflexible patterns of labour supply response. The

Table 9.15. *Performance of functional forms*

Functional form	Criterion: convenience/tractability	Behaviour for low labour supply	Flexibility of response to w
Linear	***	*	
Quadratic	**	***	**
Log-linear	**		
Semi-log	**		
Linear in shares	**	**	*
Quadratic in shares		**	***
LES	***		
CES	***		**
Quadratic direct		***	**
Quadratic indirect		***	
Indirect translog		**	***
LES where consumption involves time	***	*	***

Conventions for Table 9.15:

1. Convenience/tractability: $(n-1)$ stars are awarded if n of the relevant functional forms (labour supply, expenditure, indirect utility, direct utility) are tractable. Clearly if none were tractable the case would not have been considered
2. Behaviour for low labour supply:
   ***: Possibility of (i) $\partial l/\partial m > 0$, (ii) intersection with positive l-axis, (iii) $\partial^2 l/\partial m^2 < 0$, together with $\partial l/\partial m < 0$ for higher l
   **: two of the above
   *: one of the above
   zero : none of the above
3. Flexibility in response of l to w:
   ***: possibility of two stationary values
   **: possibility of one stationary value but which may be a maximum or minimum
   *: possibility of one stationary value which must be of a specific type
   zero : monotonic
4. Note that the award of stars in 2. and 3. above is relative to the *possibility* of certain features occurring. Whether or not they do actual occur depends on parameter values and then one should check whether other conditions such as Slutsky hold simultaneously

problem is particularly severe where participation is an issue since we are then thinking of fitting forms over an extensive range and any appeal to the idea of simplicity through local approximations becomes less persuasive. We stress in our assessment, therefore, the question of which of the functional forms appear attractive or unattractive from the point of view of our eighth category (see Table 9.1)—flexibility in response of labour supply to the wage.

The overall position is summarised in Table 9.15 where we award 'stars', *à la Michelin,* for the performance of the 12 functions with respect to the three broad categories of tractability, behaviour for low labour supply and flexibility of response to the wage. The scoring system is explained in the notes to the table.

We consider first the general performance with respect to the criteria of Table 9.15 in turn and then the specific functional forms. At least one of the relevant functional forms (labour supply, expenditure, indirect utility, direct utility) must be tractable, otherwise the case would not have been considered. This does not imply that the other forms are tractable. When we start from the labour supply function and work back to the expenditure and utility functions, then the tractability depends on the result of the integration and the possibility of inversion. When we begin with the direct or indirect function (or expenditure function) then tractability depends on the ease of solution of the maximisation problem or the first-order conditions. In general, the labour supply functions considered did not pose particular problems with integration to expenditure and indirect utility functions. The direct utility function was, however, fairly intractable except for the linear case. The specific-form utility functions (LES with and without time involved in consumption) each provided tractable expressions for all the relevant functions but the flexible utility forms (quadratic direct or indirect and indirect translog) all go with fairly inconvenient labour supply functions.

Behaviour for low labour supply which allowed a satisfactory range of possibilities was in general provided only by quadratic functions (labour supply, share, direct or indirect utility or indirect translog). The exception is 'linear-in-shares'. The explicit utility functions performed less well in this respect. Those which involve log w obviously do not allow intersection with the l-axis. The possibilities provided by the quadratic functions may go with violation of Slutsky conditions.

Flexibility in response of l to w over a range was mainly confined to the quadratic functions plus LES where consumption involves time. This last form allowed a very broad range of possibilities. Among the quadratic forms the quadratic indirect was inflexible and, outside them, the CES and linear-in-shares forms were less rigid than the other non-quadratic specifications.

We turn now to an assessment of the forms. The great virtue of the linear labour supply function is its convenience, with all relevant functions being

tractable, but it has very severe drawbacks in its inflexibility and thus the constraints imposed on the type of response permitted. The quadratic provides much greater flexibility with some loss in tractability. However, this loss is minor where quadratic terms in m do not arise, since, as we see from Table 9.4, one can easily integrate to get the expenditure function—the differential equation is linear in m. Thus the case where the labour supply is quadratic in w but linear in m seems to deserve special attention. It is a simple generalisation of the linear function, and one which is straightforward to estimate in practice, which avoids the imposition of monotonicity of labour supply with respect to the wage and which retains tractability of the important functions. Notice that the integration is especially simple if the cross-term (in wm) is absent but it remains tractable even where the term is present—as we have shown using 'erf' functions. It is the term in m^2 which raises the analytic difficulties and, in practice, it may be the least important of the extra terms in the quadratic (as compared with the linear).

The log-linear form is fairly convenient but shares with the linear and semi-log the problems of inflexibility. It is inappropriate for treating problems where low levels of labour supply are important since labour supply cannot be zero (compare its use in Burtless–Hausman, 1978, where participation is not an issue and Hausman, 1980, where participation is central and the linear form is used). The semi-log is a little less tractable than the linear without substantial advantages in flexibility. The linear-in-shares form scores reasonably across the board and it is perhaps surprising that it is little used. Its extension to quadratic terms provides greater flexibility at the cost of tractability.

All the forms which begin with the labour supply function satisfy Slutsky conditions only over a restricted range in (w, m) space.

The LES supply function is tractable but inflexible. The CES form is also tractable but provides greater flexibility. It is, however, unsatisfactory in the possibilities it permits for low levels of labour supply. Further, the flexibility that it allows over the range of labour supply comes at a cost in that simple and possibly common features of the data may dictate estimates of elasticities with very strong policy consequences which would not follow from more flexible forms. For example, if the labour supply curve is upward-sloping (even very gently so) over its range, then Figure 9.6 shows we must have $\epsilon > 1$ and $m - \bar{c} > 0$. Thus for the case of female labour supply which is commonly upward-sloping, high elasticities of substitution are essentially forced if we use the CES form. Similarly, if the data show a curvature as depicted in the fourth panel of Figure 9.6, then again we must have $m - \bar{c} < 0$ and $\epsilon > 1$ (see Brown, Levin and Ulph, 1976; and Atkinson, Stern and Gomulka, 1980, for evidence on this curvature for UK data).

The quadratic direct, indirect and the indirect translog all provide some flexibility (although for quadratic indirect this is confined to the behaviour at low levels). However, like the direct translog they generate rather inconvenient or opaque labour supply functions and, further satisfy appropriate monotonicity and concavity conditions only over certain ranges. This generates some difficulty in the interpretation of results–for example, it would be difficult to work out from the results presented by Wales and Woodland (1976), who use the indirect translog, what would be the consequences of change in estimated parameters for movements in labour supply.

The LES where consumption involves time shows strength across the board and great flexibility. It is therefore a suitable candidate for more extensive use. Estimation can be carried out as part of a labour supply/commodity demand system where it provides estimates of cross-price effects without price data and without imposing separability. Further, it may be used as a functional form for labour supply estimation alone (see Table 9.14), although in this case one may have to restrict the number of parameters for successful identification in a maximum likelihood context.

The criteria set out in Table 9.1 which do not play some role in the above discussion are number (5) on optimum income taxation and additive separability and (7) on aggregation. Only the LES and CES of the forms satisfy additive separability and we should not wish to press its importance in this context. Neither is aggregation a crucial issue in cross-section estimation when one can estimate aggregate response by adding individual responses. However, many of the forms considered here do allow some form of labour supply aggregation in that they are linear in parameters. It should be noted that the form of aggregation specified by Muellbauer (1981) is overly restrictive when one focusses on labour supply only, in that he requires aggregation of a commodity demand and labour supply system.

Our general conclusions on the functions and criteria taken together are as follows. Firstly, all of the functional forms have disadvantages either of inconvenience or of restrictions in response and one should be very cautious about placing reliance on results estimated from just one form. Secondly, the restrictions on responses at low labour supply have received little attention in the literature. Typically, the functions used allow only a narrow range of possibilities for low labour supply. It may be that we need to use different forms for low and high labour supply or splice forms together. Thirdly, we should try functions which do permit non-monotonicity of labour supply as a function of the wage. Fourthly, the flexible functional forms for the utility function are rather inconvenient and it seems better in this context to approach flexibility through the labour supply function (e.g. quadratic or quadratic-in-shares). Fifthly, if we start with the labour supply function (in flexible form or not) and integrate, then the Slutsky condition will be, in general, violated for some w and m, as also happens with flexible forms for the utility function.

In view of the above we would suggest that the quadratic, linear-in-shares, quadratic-in-shares and LES where consumption involves time have received insufficient attention in the literature, given that they have a number of attractive properties. The simple addition of w^2 to the linear supply function provides important extra possibilities in response at very little extra cost in estimation or tractability. The term in wm adds only a little extra analytic difficulty in interpretation. The CES function is quite useful in certain conditions. In particular the addition of minimum consumption levels provides a substantial extension of the possibilities. This is included in some estimations, see e.g. Brown *et al.* (1982), but not in others, e.g. Zabalza (1983). However, some simple features of the data are likely to force conclusions for the elasticity of substitution in the CES function which have strong policy implications and they should be checked against estimates from other functional forms.

9.5 Concluding remarks

Our purpose in this paper has been to set out criteria for the specification of labour supply functions and to ask how some of the functions currently used perform in relation to these criteria. Our general conclusion must be in favour of diversity of functions and great caution in drawing policy conclusions on results based on a particular form. In future applied work we hope to examine the sensitivity of policy judgements to the use of different functional forms on the same data, and this paper is, in part, a prelude to such work.

The assessment of the various forms has been set out in the previous section and will not be repeated here. We should, however, stress two points. First, any given form should not impose rigid responses; such an imposition may be an excessive and unnecessary price to pay for tractability. Secondly, greater attention should be paid to the implications of forms for low levels of labour supply.

Table 9.2. *The Functional forms*

	Form	Examples of use
Group 1: working from supply functions		
A.	Linear	Hausman (1980) and (1981*b*)
B.	Quadratic	Brown, Levin and Ulph (1976)
C.	Log-linear	Burtless and Hausman (1978)
D.	Semi-log	Heckman (1974)
F.	Quadratic in shares	
Group 2: working from utility functions		
G.	LES	Atkinson (1977); Ashenfelter (1980)
H.	CES	Stern (1976), Brown *et al.* (1982); Zabalza (1983)
I.	Quadratic direct utility	
J.	Quadratic indirect utility	
K.	Indirect translog	Wales and Woodland (1976)
L.	LES where consumption involves time	Atkinson, Stern and Gomulka (1980)

Notes:
(i) The above utility functions are assesed relative to the eight criteria presented in Table 9.1 and the results summarised in subsequent tables.
(ii) One can work from labour supply functions to utility functions or vice-versa. The first group starts with the supply function and the second the utility function.

Table 9.3. *The linear supply function*

$$l = \alpha w + \beta m + \gamma$$

1. Slutsky $\alpha - l\beta \geqslant 0$ or $(\alpha - \beta\gamma) - \alpha\beta w - \beta^2 m \geqslant 0$
If $\alpha \geqslant 0$ and $\beta \leqslant 0$ then $l \geqslant 0$ is sufficient for Slutsky
If $\alpha < 0$ and $\beta \geqslant 0$ then Slutsky is violated wherever $l \geqslant 0$

2. Linear in parameters. Additive stochastic term can imply $l < 0$.

3. $$m(w,u) = u\mathrm{e}^{-\beta w} - \frac{\alpha}{\beta} w + \frac{\alpha}{\beta^2} - \frac{\gamma}{\beta}$$

$$v(w,m) = \mathrm{e}^{\beta w}\left(m + \frac{\alpha}{\beta} w - \frac{\alpha}{\beta^2} + \frac{\gamma}{\beta}\right)$$

$$u(c,l) = \left(\frac{l-b}{\beta}\right) \exp\left\{-\left[1 + \frac{\beta(c+a)}{b-l}\right]\right\}$$

where $b = \frac{\alpha}{\beta}; a = \frac{\gamma}{\beta} - \frac{\alpha}{\beta^2}$

$$w(c,l) = \frac{(1/\beta)(l - \beta c - \gamma)}{b-l}$$

Table 9.3. (*cont.*)

4. (i) Estimation straightforward and variability across households easily incorporated
(ii) calculation of u and v is simple
(iii) derivatives of l are easily seen

5. $\frac{\partial c}{\partial m} = (\beta w + 1)$. Positive if $(\beta w + 1) > 0$, i.e. $w < -1/\beta$ (where $\beta < 0$)

$\frac{\partial l}{\partial m} = \beta$. Negative if $\beta < 0$

There is a possible conflict between normality of consumption and normality of leisure. $\frac{\partial \log w}{\partial c \partial l} \neq 0$ in general

6. $w(c,0) = -1/\alpha(\beta c + \gamma); \frac{\partial w}{\partial c} = -\beta/\alpha$ for $l = 0$.

If, e.g., $\alpha > 0, \beta < 0, \frac{\partial w}{\partial c}$ at $l = 0$ will be positive for all c. It is clear from 5. above that it is impossible for leisure to be inferior for low l and normal for high l. Supply curve intersects w-axis at $w = \frac{-(\beta m + \gamma)}{\alpha}$.

Can be positive or negative. If positive, need $\alpha > 0$ for Slutsky

7. Labour supply can be aggregated linearly

8. Inflexible response: constant derivatives, non-monotonicity in w impossible

Table 9.4. *The quadratic supply function*

$$l = \alpha w + \beta m + \lambda w^2 + \mu m^2 + \nu wm + \gamma$$

1. Slutsky $(\alpha + 2\lambda w + \nu m) - l(\beta + 2\mu m + \nu w) \geqslant 0$

2. Linear in parameters. Additive stochastic term can imply $l < 0$

3. $\mu = \nu = 0$: $m(w,u) = \mathrm{e}^{-\beta w} u + a + bw + gw^2$
$v(w,m) = \mathrm{e}^{\beta w}[m - (a + bw + gw^2)]$

where $a = -\frac{\gamma}{\beta} + \frac{\alpha}{\beta^2} - \frac{2\lambda}{\beta^3}; b = -\frac{\alpha}{\beta} + \frac{2\lambda}{\beta^2}; g = -\frac{\lambda}{\beta}$

$u(c,l) = \mathrm{e}^{\beta w}[c - wl - (a + bw + gw^2)]$
where w is a function of (c,l) given by the inverse demand function, i.e. $w(c,l)$ is the root of $\lambda w^2 + (\alpha - \beta l) w + \beta c + \gamma - l = 0$

$\mu = 0, \nu \neq 0$: $m(w,u) = (u + Q(w))\mathrm{e}^{-\beta w - 1/2 \nu w^2}$
$v(w,m) = m\, \mathrm{e}^{\beta w + 1/2 \nu w^2} - Q(w)$
where $Q(w) = \int^{w} (-\alpha w - \lambda w^2 - \gamma)\mathrm{e}^{\beta w + 1/2 \nu w^2} \mathrm{d}w$

Table 9.4. *(cont.)*

Q is essentially in closed form using the standard function, erf $(z) = \int^{z} e^{-t^2} dt$. Thus if we put $t = w + \beta/\nu$ we have to integrate terms in $te^{1/2\,\nu t^2}$ and $t^2 e^{1/2\,\nu t^2}$. The integral of the first is $(2/\nu)e^{1/2\,\nu t^2}$ and the second is obtained, using erf on integrating $t \cdot te^{1/2\,\nu t^2}$ by parts
$u(c, l) = (c - \hat{w}l)e^{\beta\hat{w} + 1/2\,\nu\hat{w}^2} - Q(\hat{w})$ where $\hat{w}$ satisfies
$w^2(\lambda - \nu l) + (\alpha + c\nu - \beta l)\, w + \beta c + \gamma - l = 0$
$w(c, l)$ from root of $(\lambda - \nu l)\, w^2 + (\alpha - \beta l + \nu c)\, w + \beta c + \gamma - l = 0$
μ and $\nu \neq 0$ then $v(w, m)$ can be derived from Schrödinger's equation (see Hausman, 1981).

(i) Estimation straightforward and variability across households easily seen
(ii) Calculation of u and v particularly straightforward when $\mu = \nu = 0$
(iii) Derivatives of l easily calculated but less transparent for cases other than $\mu = \nu = 0$

5. $$\frac{\partial c}{\partial m} = \beta w + \nu w^2 + 2\mu m w + 1$$

$$\frac{\partial l}{\partial m} = \beta + \nu w + 2\mu m$$

For $\mu = \nu = 0$ we have the same form for the derivatives wrt m as the linear case

$$\frac{\partial \log w}{\partial c \partial l} \neq 0 \text{ in general}$$

6. $w(c, 0)$ from $\lambda w^2 + (\alpha + \nu c)\, w + \beta c + \gamma + \mu c^2 = 0$

At $l = 0$ $\dfrac{\partial w}{\partial c} = -\dfrac{\nu w + 2\mu c + \beta}{2\lambda w + \nu c + \alpha}$

and for $\mu = \nu = 0, = -\dfrac{\beta}{\alpha + 2\lambda w}$

Hence, in this case, where it is differentiable w is monotonic in c; it is increasing (decreasing) as β is negative (positive). Supply curve can intersect positive w-axis, zero, one or two times but at the axis Slutsky holds only where the gradient is positive

7. Linear aggregation is not possible

8. If $\lambda \neq 0$ we have the possibility of a flexible response of l as a function of w. The labour supply curve can be forward- or backward-bending.

Table 9.5. *The log-linear function*

$l = kw^{\alpha}m^{\beta}$; $\log l = \alpha \log w + \beta \log m + \log k$ (requires $m > 0$)

1. Slutsky $\dfrac{\beta wl}{m} \leqslant \alpha$ or $\beta k w^{\alpha+1} m^{\beta-1} \leqslant \alpha$

– satisfied if $\alpha \geqslant 0, \beta \leqslant 0$

2. Linear in parameters in logarithmic version. Difficulties arise if m is zero or negative

3. $$m(w, u) = \frac{u - kw^{1+\alpha}}{1 + \alpha}(1 - \beta)^{1/1-\beta}$$

$$v(w, m) = \frac{kw^{1+\alpha}}{1 + \alpha} + \frac{m^{1-\beta}}{1 - \beta}$$

$$u(c, l) = \frac{kw^{1+\alpha}}{1 + \alpha} + \frac{(c - wl)^{1-\beta}}{1 - \beta}$$

where w is a function of (c, l) given by the inverse demand function, i.e. $w(c, l)$ is the root of $l = kw^{\alpha}(c - wl)^{\beta}$

4. (i) Estimation straightforward and variability across households easily incorporated
(ii) Calculation of u and v is simple
(iii) Derivatives of l are easily seen

5. $\dfrac{\partial c}{\partial m} = \dfrac{\beta wl}{m} + 1$. Positive if $\beta > 0$ or $\beta < 0$ and $\dfrac{wl}{m} < -\dfrac{1}{\beta}$

$\dfrac{\partial l}{\partial m} = \dfrac{\beta l}{m}$. Negative if $\beta < 0$

$$\frac{\partial \log w}{\partial c \partial l} \neq 0 \text{ in general}$$

6. $l = 0$ is impossible

7. Aggregation is possible using geometric means

8. Inflexible response: constant elasticities, non-monotonicity impossible

Table 9.6. *Semi-log supply function*

$l = \alpha \log w + \beta m + \gamma$

1. Slutsky: $\alpha \geqslant \beta wl$

2. Linear in parameters. Additive stochastic term can imply $l < 0$

3. $$m(w, u) = e^{-\beta w}u - \frac{\gamma}{\beta} - \frac{\alpha}{\beta}\log w + \frac{\alpha}{\beta}e^{-\beta w}Ei(\beta w)$$

where $Ei(x)$ is the standard exponential integral $\int_{-\infty}^{x}(e^t/t)dt$ (see Abramowitz and Stegun (1965), p. 228)

Table 9.6. (*cont.*)

$$v(w,m) = \frac{e^{\beta w}}{\beta}(m\beta + \gamma + \alpha \log w) - \frac{\alpha}{\beta} Ei(\beta w)$$

$$u(c,l) = \frac{e^{\beta w}}{\beta}(\beta(c - wl) + \gamma + \alpha \log w) - \frac{\alpha}{\beta} Ei(\beta w)$$

where w is a function of (c, l) given by the inverse demand function, i.e. $w(c, l)$ is the root of $\alpha \log w - \beta wl = l - \beta c - \gamma$

4. (i) Estimation straightforward and variability across households easily incorported
(ii) Caclulation of $v(\ )$ and $m(\ )$ is straightforward but not $u(\ )$
(iii) Derivatives of l easily seen

5. $\frac{\partial l}{\partial m} = \beta; \frac{\partial c}{\partial m} = \beta w + 1$

$\frac{\partial \log w}{\partial c \partial l} \neq 0$ in general

6. $w(c, 0) = \exp\left\{-\left(\frac{\beta c + \gamma}{\alpha}\right)\right\}; \frac{\partial w}{\partial c} = -\frac{\beta}{\alpha}\exp\left\{-\left(\frac{\beta c + \gamma}{\alpha}\right)\right\}$. Positive if $\beta < 0$ and $\alpha > 0$

7. Labour supply can be aggregated using geometric mean of wage

8. Inflexible wage response

Table 9.7. *Share linear in logarithms*

$$\frac{wl}{m} = \alpha \log w + \beta \log m + \gamma \text{ (requires } m > 0)$$

1. Slutsky $\frac{-(1+\beta)}{2} - \frac{1}{2}\sqrt{((1+\beta)^2 + 4\alpha)} \leqslant \frac{wl}{m} \leqslant \frac{-(1+\beta)}{2} +$
$\frac{1}{2}\sqrt{((1+\beta^2) + 4\alpha)}$
where $(1+\beta)^2 \geqslant -4\alpha$

2. Linear in parameters. Additive stochastic term can imply $l < 0$. Difficulties arise if m is zero or negative

3. Cost and indirect utility functions from putting $w \rightarrow \log w$ and $m \rightarrow \log m$ in linear case (see Table 9.3)

$$\log m(w, u) = uw^{-\beta} - \frac{\alpha}{\beta}\log w + \frac{\alpha}{\beta^2} - \frac{\gamma}{\beta}$$

$$v(w, m) = w^{\beta}(\log m + \frac{\alpha}{\beta}\log w - \frac{\alpha}{\beta^2} + \frac{\gamma}{\beta}$$

Table 9.7. (*cont.*)

$$u(c,l) = \frac{w^{\beta}}{\beta^2}\left(\frac{\beta wl}{(c - wl)} - \alpha\right)$$

where w is a function of (c, l) given by the inverse demand function, i.e. $w(c, l)$ is the root of $(wl)/(c - wl) = \alpha \log w + \beta \log (c - wl) + \gamma$

4. (i) Estimation straightforward and variability across households easily incorporated
(ii) Calculation of v simple
(iii) $\frac{w}{l}\frac{\partial l}{\partial w} = \frac{\alpha m}{wl} - 1; \frac{m}{l}\frac{\partial l}{\partial m} = \frac{\beta m}{wl} + 1$. Elasticities straightforward

5. $\frac{\partial c}{\partial m} = (1 + \beta) + \frac{wl}{m} > 0$ if $1 + \frac{wl}{m} > -\beta$

$\frac{\partial l}{\partial m} = \frac{l}{m}\left(\frac{\beta m}{wl} + 1\right) < 0$ if $m > 0$ and

$1 < -\frac{\beta m}{wl}$

$\frac{\partial \log w}{\partial c \partial l} \neq 0$ in general

6. $w(c, 0) = e^{-\gamma/\alpha}c^{-\beta/\alpha}$ which increases with c if $\beta/\alpha < 0$
Supply curve intersects w-axis at $w = e^{-\gamma/\alpha}m^{-\beta/\alpha} > 0$

7. Linear aggregation only in the sense of wl/m as a function of the geometric means of w and m

8. $\frac{\partial l}{\partial w} = \frac{\alpha m}{w^2} - \frac{l}{w} > 0$ at $l = 0$ if $\alpha > 0$

$\frac{w \partial l}{l \partial w} = \frac{\alpha}{\alpha \log w + \beta \log m + \gamma} \rightarrow -1$ as $w \rightarrow \infty$ if $\alpha > 0$

$\alpha > 0$

$\frac{w \partial l}{l \partial w} = 0$ where $\frac{wl}{m} = \alpha$ or $\alpha \log w + \beta \log m + \gamma\beta \log m + \gamma = \alpha$

$\frac{\partial l}{\partial w} < 0$ for all positive l if $\alpha < 0$ (for $m > 0$)

$\lim_{w \rightarrow \infty} l = 0$ $\lim_{w \rightarrow 0} l = -\infty$ (for $\alpha > 0$)
$+\infty$ (for $\alpha < 0$)

Table 9.8. *Share quadratic in logarithms*

$$\frac{wl}{m} = \alpha \log w + \beta \log m + \lambda(\log w)^2 + \mu(\log m)^2 + \nu \log w \log m + \gamma$$

1. Slutsky from $\left(\frac{w}{l}\frac{\partial l}{\partial w}\right) - \left(\frac{wl}{m}\right)\left(\frac{m}{l}\frac{\partial l}{\partial m}\right) \geqslant 0$

2. Linear in parameters. additive stochastic term can imply $l < 0$. Difficulties arise if m is zero or negative

3. The functional forms for the cost and indirect utility functions are derived from Table 9.4 by putting $w \to \log w$ and $m \to \log m$ just as the entries in Table 9.7 were derived from Table 9.3. We give the functional forms for $\mu = \nu = 0$ as an illustration:
$\log m(w, u) = uw^{-\beta} + a + b \log w + g(\log w)^2$
$v(w, m) = w^{\beta}[\log m - (a + b \log w + g(\log w)^2)]$
$$u(c, l) = \frac{w^{\beta}}{\beta^2}\left(\frac{\beta wl}{c - wl} + \beta b + 2\, g\beta \log w\right)$$
where w is a function of (c, l) given by the inverse demand function, i.e. $w(c, l)$ is the root of
$$\beta \log(c - wl) + \alpha \log w - \lambda(\log w)^2 + \gamma = \frac{wl}{c - wl}$$

4. $$\frac{w}{l}\frac{\partial l}{\partial w} = \frac{\alpha m}{wl} + \frac{2m\, \lambda \log w}{lw} - 1$$
$$\frac{m}{l}\frac{\partial l}{\partial m} = \frac{\beta m}{wl} + 1$$

5–7. Omitted

8. Sketching the labour supply curve:
$w \to \infty\; l \to 0$, $w \to 0\; l \to \infty$ } in each case the limiting sign of l is that of λ
$$\frac{w}{l}\frac{\partial l}{\partial w} = \frac{\alpha + 2\lambda \log w}{\alpha \log w + \beta \log m + \gamma + \lambda(\log w)^2} - 1$$
Putting $x = \log w$, $\beta \log m + \gamma = \delta$ we have
$$\frac{\partial l}{\partial w} \gtreqless 0 \text{ as } \alpha + 2\lambda x \gtreqless \lambda x^2 + \alpha x + \delta$$
and where we require for $l \geqslant 0$, $\lambda x^2 + \alpha x + \delta \geqslant 0$

Table 9.9. *Linear expenditure system*

$$u(c, l) = \alpha_1 \log (c - \bar{c}) + \alpha_2 \log (T - l)$$
$$wl = \alpha_1 wT - \alpha_2 (m - \bar{c})$$
$$\alpha_1, \alpha_2 > 0 \quad \text{and} \quad \alpha_1 + \alpha_2 = 1$$

1. Slutsky satisfied

2. Linear in parameters. Additive stochastic term can imply $l < 0$

3. $m(w, u) = uw^{\alpha_2} + \bar{c} - wT$
$v(w, m) = w^{-\alpha_2}(m + wT - \bar{c})$
$$w(c, l) = \frac{\alpha_2 (c - \bar{c})}{\alpha_1 (T - l)}$$

4. (i) Estimation straightforward and variability across households easily incorporated
(ii) Derivatives of l are easily seen:
$$\frac{\partial l}{\partial w} = \frac{\alpha_1 T - l}{w};\ \frac{\partial l}{\partial m} = -\frac{\alpha_2}{w}$$

5. $$\frac{\partial c}{\partial m} = \alpha_1 > 0$$
$$\frac{\partial l}{\partial m} = -\frac{\alpha_2}{w} < 0$$
$$\frac{\partial \log w}{\partial c \partial l} = 0. \text{ Obviously additively separable}$$

6. $w(c, 0) = \frac{\alpha_2}{\alpha_1 T}(c - \bar{c})$ and $\frac{\partial w}{\partial c} > 0$ for $l = 0$
Supply curve intersects w-axis at $\frac{\alpha_2 (m - \bar{c})}{\alpha_1 T}$ if $m > \bar{c}$

7. Linear aggregation of earnings as function of w and m

8. Inflexible response: either forward-sloping if $m > \bar{c}$ or backward if $m < \bar{c}$

Table 9.10. *CES*

$$u(c, l) = [\alpha(T - l)^{-\mu} + (1 - \alpha)(c - \bar{c})^{-\mu}]^{-1/\mu}$$
$$T - l = \frac{m + wT - \bar{c}}{w + kw^{\epsilon}} \qquad k = \left(\frac{1 - \alpha}{\alpha}\right)^{\epsilon}, \epsilon = \frac{1}{1 + \mu}$$

Table 9.10. *(cont.)*

$$\frac{(c-\bar{c})}{T-l} = kw^{\epsilon} \qquad \mu > -1,\ \epsilon > 0$$

1. Slutsky satisfied for $\epsilon > 0$

2. $\log\left(\frac{c-\bar{c}}{T-l}\right) = \epsilon \log w + \log k$ is linear for estimation if $\bar{c}$ and T are known although they will clearly not be in general

3. $m(w,u) = wu[1 + kw^{\epsilon-1}]^{1/(1-\epsilon)} - wT + \bar{c}$

$$v(w,m) = \frac{1}{w}(m + wT - \bar{c})[1 + kw^{\epsilon-1}]^{1/(\epsilon-1)}$$

$$w(c,l) = \frac{\alpha}{(1-\alpha)}\left(\frac{c-\bar{c}}{T-l}\right)^{-1/\epsilon}$$

4. (i) Estimation straightforward as linear regression if T and $\bar{c}$ known or assumed–otherwise non-linear methods necessary
(ii) Household characteristics are naturally introduced in $\bar{c}$ and T but this involves non-linearities
(iii) Elasticity of substitution between $(c-\bar{c})$ and $(T-l)$ is easily seen but $\partial l/\partial w$ is less transparent

5. $\frac{\partial l}{\partial m} = -\frac{1}{w + kw^{\epsilon}} < 0 \qquad \frac{\partial \log w}{\partial c \partial l} = 0$; obviously additively separable

$$\frac{\partial c}{\partial m} = \frac{kw^{\epsilon}}{w + kw^{\epsilon}} > 0$$

6. $w(c,0) = \frac{\alpha}{(1-\alpha)T}(c-\bar{c})^{1/\epsilon}; \qquad \frac{\partial w}{\partial c} > 0 \qquad$ for $\qquad l = 0$

Supply curve intersects w-axis at $\frac{\alpha}{(1-\alpha)T}(m-\bar{c})^{1/\epsilon}$ if $m > \bar{c}$. If $\bar{c} > m$, $l = 0$ impossible

7. Linear aggregation is not possible

8. $\frac{\partial l}{\partial w} \gtrless 0$ as $(1-\epsilon)kTw^{\epsilon} \lesseqgtr (m-\bar{c})(1 + \epsilon k w^{\epsilon-1})$

if $m - \bar{c} > 0$ and $0 < \epsilon < 1$ the rhs decreases in w and the lhs increases and for small w, $\partial l/\partial w > 0$, for w large $\partial l/\partial w < 0$ and there is a unique w with $\partial l/\partial w = 0$

if $m - \bar{c} < 0$ and $0 < \epsilon < 1$ then $\frac{\partial l}{\partial w} < 0$

Table 9.10. *(cont.)*

if $m - \bar{c} > 0$ and $\epsilon > 1$ then $\frac{\partial l}{\partial w} > 0$

if $m - \bar{c} < 0$ and $\epsilon > 1$ $\frac{\partial l}{\partial w} > 0$ for large w and < 0 for small w

Table 9.11. *Quadratic direct utility*

$$u(c,l) = \alpha c^2 + \beta l^2 + \gamma cl + \delta l + \epsilon c$$

$$l = \frac{2\alpha m w + \epsilon w + \gamma m + \delta}{2(\alpha w^2 + \gamma w + \beta)};\ c = wl + m$$

1. Concavity in $(c, -l)$ requires $\begin{bmatrix} 2\alpha & -\gamma \\ -\gamma & 2\beta \end{bmatrix}$ to be negative definite:
$\alpha, \beta < 0$ and $4\alpha\beta - \gamma^2 > 0$
If $\alpha < 0$ then u is monotonic increasing in c if $\gamma l + \epsilon > -2\alpha c$
If $\beta < 0$ then u is monotonic decreasing in l if $\gamma c + \delta < -2\beta l$

Slutsky: $[(\partial l/\partial w) - (l \partial l/\partial m)] \geqslant 0$. If $\alpha = 0$ then require $\gamma w + \beta \geqslant 0$ and $\epsilon - 3\gamma l \geqslant 0$ or $\gamma w + \beta \leqslant 0$ and $\epsilon - 3\gamma l \leqslant 0$
If $\alpha \neq 0$ then expression is messy

2. Linear if multiplied through by denominator in expression for l
If $\alpha = 0$ then have linear expression in wl, l, w, m. If $\alpha = \gamma = 0$ then l is linear in w.

3. Indirect utility function straightforward on substituting for l and c functions in direct function, although inelegant. Expenditure function less straightforward since indirect utility function is quartic in m (quadratic if $\alpha = 0$). If $\alpha = \gamma = 0$ then

$$w(c,l) = \frac{\gamma c + \delta - 2l\beta}{3\gamma l - \epsilon} \qquad m(w,u) = u - \frac{\epsilon w^2}{4\beta} - \frac{\delta w}{2\beta}$$

4. (i) Estimation straightforward and variability across households easily incorporated
(ii) Calculation of v straightforward (although not m)
(iii) Derivative of l wrt w is messy unless $\alpha = 0$. In this case

$$\frac{\partial l}{\partial w} = \frac{-\gamma l}{\gamma w + \beta} + \frac{\epsilon}{2(\gamma w + \beta)};\ \frac{\partial l}{\partial m} = \frac{2\alpha w + \gamma}{2(\alpha w^2 + \gamma w + \beta)}$$

5. Where $\alpha = 0$, $\frac{\partial l}{\partial m} < 0$ if $\gamma > 0$ and $\gamma w + \beta < 0$ or $\gamma < 0$ and $\gamma w + \beta > 0$

Table 9.11. *(cont.)*

$$\frac{\partial c}{\partial m} = \frac{\gamma w}{2(\gamma w + \beta)} + 1 > 0 \text{ if } \gamma w + \beta < 0 \text{ and } 3\gamma w + 2\beta < 0$$

$$\text{or if } \gamma w + \beta > 0 \text{ and } 3\gamma w + 2\beta > 0$$

Additive separability if $\gamma = 0$

6. $$w(c, 0) = -\frac{(\gamma c + \delta)}{2\alpha c + \epsilon}; \frac{\partial w}{\partial c} = \frac{2\alpha\delta - \gamma\epsilon}{(2\alpha c + \epsilon)^2}$$

which is $\gtrless 0$ as $2\alpha\delta - \gamma\epsilon \gtrless 0$
Supply curve intersects w-axis where $w = -(\gamma m + \delta)/(2\alpha m + \epsilon)$. Can be positive or negative

7. Linear aggregation not possible in general. If $\alpha = 0$ then it is possible using means of (wl), l, w and m

8. Sign of $\partial l/\partial w$ can change with w. If $\alpha = 0$ then

$$\frac{\partial l}{\partial w} > 0 \text{ if } (\gamma w + \beta) > 0 \text{ and } -2\gamma l + \epsilon > 0$$

$$\text{or if } (\gamma w + \beta) < 0 \text{ and } -2\gamma l + \epsilon < 0$$

Table 9.12. *Quadratic indirect utility*

$$v(w, m) = \alpha m^2 + \beta w^2 + \gamma wm + \delta w + \epsilon m$$

$$l = \frac{2\beta w + \gamma m + \delta}{2\alpha m + \gamma w + \epsilon} \text{ (note: same as Table 9.11 if } \alpha = 0)$$

1. $v(w, m)$ increases in w if $2\beta w + \gamma m + \delta > 0$ and in m if $2\alpha m + \gamma w + \epsilon > 0$

Convex if $\begin{bmatrix} 2\alpha & \gamma \\ \gamma & 2\beta \end{bmatrix}$ positive definite, i.e. if: $\alpha, \beta > 0; 4\alpha\beta - \gamma^2 > 0$

2. Can be estimated linearly if multiply through by denominator

3. $u(c, l) = \alpha(c - wl)^2 + \beta w^2 + \gamma w(c - wl) + \delta w + \epsilon(c - wl)$
where w is given by $w(c, l)$ below
$m(w, u) = (u - \beta w^2 - \delta w)/(\gamma w + \epsilon)$ if $\alpha = 0$ otherwise from the root of the quadratic
$\alpha m^2 + (\gamma w + \epsilon)m + \beta w^2 + \delta w - u = 0$

$$w(c, l) = \frac{2\alpha cl + \epsilon l - \gamma c - \delta}{2\beta - 2\gamma l + 2\alpha l^2}$$

4. (i) Estimation linear in ml, wl, w, l on multiplying by denominator in l
(ii) Calculation of $v(w, m)$, $m(w, u)$ (if $\alpha = 0$) straightforward

$$\frac{\partial l}{\partial w} = \frac{(4\beta\alpha - \gamma^2)m + 2\beta\epsilon - \gamma\delta}{(2\alpha m + \gamma w + \epsilon)^2}; \frac{\partial l}{\partial m} = \frac{(\gamma^2 - 4\alpha\beta)w + \gamma\epsilon - 2\alpha\delta}{(2\alpha m + \gamma w + \epsilon)^2}$$

Table 9.12. *(cont.)*

(iii) Derivatives not particularly transparent

5. $$\frac{\partial l}{\partial m} < 0 \text{ if } w > \frac{-2\alpha\delta + \gamma\epsilon}{4\alpha\beta - \gamma^2}$$

Not additively separable unless $\alpha = \gamma = 0$

6. $$w(c, 0) = \frac{-(\gamma c + \delta)}{2\beta}; \frac{\partial w}{\partial c} = -\frac{\gamma}{2\beta}$$

Supply curve intersects w-axis where $w = -(\gamma m + \delta)/2\beta$. Can be positive or negative

7. Linear aggregation not possible in general

8. Sign of $\partial l/\partial w$ independent of w. It can change with m

Table 9.13. *Indirect translog*

$$v(w, m) = \alpha(\log m)^2 + \beta(\log w)^2 + \gamma \log w \log m + \delta \log w + \epsilon \log m$$

$$\frac{wl}{m} = \frac{2\beta \log w + \gamma \log m + \delta}{2\alpha \log m + \gamma \log w + \epsilon}$$

1. $v(w, m)$ increases in w if $A = 2\beta \log w + \gamma \log m + \delta > 0$ and in m if $B = 2\alpha \log m + \gamma \log w + \epsilon > 0$
Convex if $2\beta > A$, $2\alpha > B$ and $(2\beta - A)(2\alpha - B) > \gamma^2$

2. Can be estimated linearly if multiply through by denominator in expression for wl/m. Difficulties arise if m is zero or negative

3. $u(c, l)$ from substituting from inverse demand function $w(c, l)$ in $v(w, c - wl)$: messy
$m(w, u)$ cannot be obtained simply since $v(w, m)$ is quadratic in $\log m$
$w(c, l)$ on substituting $c = wl + m$ in share equation: messy

4. (i) Estimation linear if multiply by denominator in share equation
(ii) Calculation of $v(w, m)$ and $m(w, u)$ and derivatives of l are messy

5. Derivatives of c and l wrt m are messy. Not additively separable

6. $w(c, 0) = kc^{-\gamma/2\beta}$ where $k = e^{-\delta/2\beta}$. Increasing in c if $\gamma/2\beta < 0$
Supply curve intersects w-axis where $w = km^{-\gamma/2\beta}$

7. Linear aggregation not possible

8. Sign of $\partial l/\partial w$ can change with w

Note: direct translog
$u(c, l) = \alpha (\log c)^2 + \beta (\log l)^2 + \gamma \log c \log l + \delta \log l + \epsilon \log c$ gives intractable supply function

Table 9.14. *LES where consumption activity involves time*

$$u(c) = \prod_{i=1}^{n} (c_i - \gamma_i)^{\beta_i} \qquad \beta_i \geqslant 0 \text{ and } \sum_{i=1}^{n} \beta_i = 1$$

Constraint on time: $t \cdot c \leqslant T - l$; on money: $p \cdot c \leqslant m + wl \cdot c$, p and t are vectors. The level of activity t_i is the (per unit) time input into activity i; and p_i is its money cost (per unit). c_n is pure leisure: $t_n = 1, p_n = 0$. Labour l, does not enter utility function directly.

$$l = T' - (m' + wT') \sum_{i=1}^{n} \beta_i \frac{t_i}{p_i + wt_i} \qquad \left(\begin{array}{l} T' = T - t \cdot \gamma > 0 \\ m' = m - p \cdot \gamma \\ 0 \leqslant l \leqslant T' \text{ if } t_i \geqslant 0 \\ m' + wT' \geqslant 0 \end{array} \right.$$

[Note that demand for consumption activity satisfies $(p_i + wt_i)(c_i - \gamma_i) = \beta_i(m' + wT')$. This is analogous to LES with effective price of the activity $p_i + wt_i$ which must be positive for the validity of the model.]

1. Slutsky satisfied (see Atkinson and Stern, 1979)

2. Non-linearities arise because t_i (or t_i/p_i) is unknown and must be estimated

3. $$m(p, w, u) + wT = (p + wT) \cdot \gamma + u \prod_{i=1}^{n} (p_i + wt_i)^{\beta_i}$$

$$v(p, w, m) = (m' + wT') \prod_{i=1}^{n} (p_i + wt_i)^{-\beta_i}$$

$w(c, l)$ on substituting $m = p \cdot c - wl$ in labour supply function but fairly messy

4. Estimation non-linear (see e.g. Atkinson, Stern and Gomulka, 1980)

$$\frac{\partial l}{\partial w} = \sum_{i=1}^{n} \frac{\beta_i t_i (m' t_i - p_i T')}{(p_i + wt_i)^2} \qquad \frac{\partial l}{\partial m} = -\sum_{i=1}^{n} \frac{\beta_i t_i}{(p_i + wt_i)}$$

5. $$\frac{\partial c_i}{\partial m} = \frac{\beta_i}{(p_i + wt_i)} > 0; \quad \frac{\partial l}{\partial m} = -\sum_{i=1}^{n} \frac{\beta_i t_i}{(p_i + wt_i)} < 0 \text{ (if all } t_i \geqslant 0)$$

Additive separability in activities but not in goods and labour

6. $w(c, 0)$ satisfies

$$\frac{T'}{p \cdot (c - \gamma) + wT'} = \sum_{i=1}^{n} \frac{\beta_i t_i}{p_i + wt_i}$$

Differentiating wrt c_j one can show $\partial w / \partial c_j > 0$

$\lim_{w \to 0} l$ in labour supply curve is $T' - m' \sum_{i=1}^{n} t_i \frac{\beta_i}{p_i}$. If $\beta_n > 0$ then this is negative (since $p_n = 0$) if $m' > 0$ $(m > p \cdot \gamma)$ and the supply curve intersects the w-axis at positive w. If $m' < 0$ $t_i \geqslant 0$ and then l reaches the upper bound T' for positive w. (see 8. below and text.)

7. Linear aggregation not possible

Table 9.14. *(cont.)*

8. Highly flexible in that there are many possibilities. To sketch the curve we not note the following. See Figure 9.9 for examples.

(i) $\partial l / \partial w > 0$ where $l = 0$ (Slutsky). Hence if there exists w_0 for which $l(w) = 0$ then l is zero for $w \leqslant w_0$ and strictly positive for $w > w_0$

(ii) If $t_i \geqslant 0$ and $\frac{m'}{T'} > \sum_{i=1}^{n} \frac{\beta_i p_i}{t_i}$ then l is zero for all w

then l is zero for all w $(w_0 = \infty)$. Otherwise l takes positive values for some w

(iiia) For $m' > 0$:
if

$$\frac{T'}{m'} < \sum_{i=1}^{n} \beta_i \frac{t_i}{p_i}$$

then $w_0 > 0$, i.e. $l = 0$ at some positive wage, and if

$$\frac{T'}{m'} \geqslant \sum_{i=1}^{n} \beta_i \frac{t_i}{p_i}$$

then $w_0 = 0$, i.e. there is no positive wage for which $l = 0$
Note that if $\beta_n > 0$ then the former case is relevant (since $p_n = 0$).
(iiib) If $m' < 0$ then l is equal to T' for $w = -m'/T'$. The model is invalid for lower values of w. Further, l is never zero. If $t_i \geqslant 0$ then T' is the maximum possible for l

(iv) $t_i \geqslant 0$: then $\lim_{w \to \infty} l = T' \, (1 - \Sigma_{t_i \neq 0} \, \beta_i)$ (where the summation is over those i for which $t_i > 0$)

$= 0$ if all $t_i > 0$

Some $t_i < 0$: then $l \to \infty$ as $w \to w^*$ where w^* is the minimum of w_j^* where $t_j < 0$ and $w_j^* = -p_j/t_j$. For $w > w^*$ the model is invalid

$$\text{(v)} \quad \frac{\partial l}{\partial w} = \frac{1}{w^2} \left[\beta_n m' + \sum_{i=1}^{n-1} \frac{\beta_i t_i (m' t_i - p_i T')}{(p_i/w + t_i)^2} \right] = \sum_{i=1}^{n} \frac{\beta_i t_i (m' t_i - p_i T')}{(p_i + wt_i)^2}$$

Sign will depend on signs of $t_i(m' t_i - p_i T')$ which is positive if $t_i < 0$ see (b) below–and negative if $m' < 0$ and $t_i > 0$). Some cases:
(a) $t_i > 0$ and $m' t_i - p_i T' < 0$ all $i = 1, 2, \ldots (n - 1)$ then if $m' > 0$, $\partial l / \partial w$ is positive for small w; is zero at one and only one value and negative for large w, if $m' < 0$ then $\partial l / \partial w < 0$ (note $m' < 0$, $t_i > 0$ implies $m' t_i - p_i T < 0$)
(b) $t_i < 0$ all $i = 1, 2, \ldots (n - 1)$
Only one turning point is possible since $t_i(m' t_i - p_i T') > 0$ whenever $t_i < 0$ (using $-m'/T' < w^*$)
(c) Some t_i positive, some negative but $t_i(m' t_i - p_i T')$ always positive:

$$\frac{\partial l}{\partial w} > 0$$

(d) $t_i(m' t_i - p_i T')$ positive for some i, negative for others, then more than one turning point is possible.

NOTE

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